VIBRATORY MOTION

AND

SOUND

BY

J. D. EVERETT, M.A., D.C.L., F.R.S., F.R.S.E.

PROFESSOR OF NATURAL PHILOSOPHY IN THE
QUEEN'S COLLEGE, BELFAST

LONDON
LONGMANS, GREEN, AND CO.
1882

PREFACE.

THE PRESENT TREATISE is intended to open up the subject of Vibratory Motion to students who have mastered the elements of dynamics.

The key to the whole subject is Simple Harmonic motion, to which accordingly a large amount of space has been devoted. Its definition and leading properties are discussed in Chapter I.

Chapter III. discusses the composition of two Simple Harmonic motions of the same period, and contains some novelties in the shape of simple geometric proofs of propositions usually established by trigonometry (see especially §§ 24–28). Chapter II. has cleared the way by some explanations regarding the general subject of composition of motions.

These first three chapters lead straight to the geometry of Simple Harmonic waves, which is accordingly discussed in Chapter IV. The proofs of some of the most important properties of waves are given in duplicate,

G

CONTENTS.

CHAPTER I.

SIMPLE HARMONIC VIBRATION.

CHAPTER II.

COMPOSITION OF MOTIONS OF TRANSLATION.

CHAPTER III.

COMPOSITION OF VIBRATIONS OF THE SAME PERIOD.

CHAPTER IV.

WAVES.

CHAPTER V.

COMPOSITION OF TWO S.H. UNDULATIONS OF EQUAL WAVE-LENGTH.

CHAPTER VI.

COMPOSITION OF TWO S.H. MOTIONS OF DIFFERENT PERIODS.

CHAPTER VII.

MECHANICAL ILLUSTRATIONS.

CHAPTER VIII.

PROPAGATION OF SONOROUS UNDULATIONS.

CHAPTER IX

REFLECTION OF SONOROUS UNDULATIONS

CHAPTER X

DYNAMICAL INVESTIGATION

CHAPTER XI.

ENERGY OF VIBRATIONS

CHAPTER XII.

SIMPLE AND COMPOUND TONES

CHAPTER XIII

MUSICAL INTERVALS.

VIBRATORY MOTION

AND

SOUND.

CHAPTER I.

SIMPLE HARMONIC VIBRATION.

1. WHEN the prongs of a tuning-fork are squeezed between the fingers and suddenly released, they spring back not only to their original position, but to a nearly equal distance on the other side, and swing backwards and forwards a great number of times before they finally come to rest. This is an example of *vibration.*

The time occupied in swinging from one side to the other and back again is called the *periodic time*, or the *period of vibration*, or simply the *period*, and the distance that any particle of the fork travels, first to one side and then to the other side of its position of equilibrium, is called the *amplitude* of vibration for this particle.

2. A tuning-fork, when well started, usually makes several thousand vibrations before coming to rest. Their amplitudes gradually decrease, and hence the sound

emitted becomes fainter; but the periodic time remains sensibly constant, and hence the pitch does not sensibly alter. The pitch of a sound depends only on the periodic time of the vibrations which are its physical cause. Any two bodies, however different in character, if they vibrate with equal periodic times, produce sounds of the same pitch. And it is a general law that unless a vibrating body be very widely distorted from its position of equilibrium, its periodic time, and therefore its pitch, are independent of the amplitude of vibration. Most of our musical instruments give sounds varying greatly in loudness according to the force employed in producing them; but though this force influences the loudness, it does not influence the pitch, or music would be well nigh impossible.

3. This constancy of pitch is closely connected with the following law of elastic resistance. When an elastic body is distorted from its natural form or size, the force required to distort it, or, what is equal and opposite to this, the force of restitution exerted by the body, is directly proportional to the amount of distortion. For example, if we compare the force with which a tuning fork must be squeezed to make its prongs approach by $\frac{1}{50}$th of an inch with that required to make them approach by $\frac{1}{100}$th of an inch, we shall find the former to be precisely double of the latter. When the fork is vibrating, the forces of elasticity are always urging it towards its position of equilibrium. They vanish for an instant

when it is passing through this position, they then gradually increase as it departs further from this position, and attain their maximum in the position of greatest displacement. Thus far they have been opposing and gradually destroying its motion, until, in the extreme position, it comes for an instant to rest. As it returns to the position of equilibrium they go through the same values again in backward order, and restore to it the velocity which they previously destroyed, but in the opposite direction. Similar action occurs on the other side of the position of equilibrium, and the whole motion of the fork can thus be divided into four equal parts which are reversed copies of each other.

4. The proportionality of the elastic force called out by displacement to the displacement itself is thus a fundamental law of the vibratory motions which give rise to musical sound; and we shall commence our analysis of vibratory motion by discussing the simplest conceivable case—the case of *a particle vibrating in a straight line under the action of a force which urges it towards the middle point of its path, and varies directly as the distance of the particle from this point.* The motion of a particle under these conditions is called *Simple Harmonic Motion.*

SIMPLE HARMONIC MOTION.

5. We know that a particle moving with uniform velocity in a circle is acted on by a constant force directed towards the centre

Let P be the particle, and C the centre. Draw P X and P Y perpendiculars on two fixed diameters A A′, B B′. The force acting on P can be resolved into two components, one parallel to X C, and the other to Y C, and these are represented by the lines X C and Y C on the same scale on which the whole force is represented by P C. Hence the component force parallel to A A′ is proportional to X C But the component motion of P parallel to A A′ is the same as the whole motion of X Hence the point X moves as if urged towards C by a force proportional to X C. The point X therefore executes simple harmonic motion.

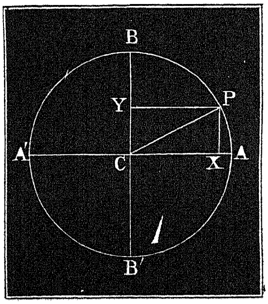

FIG 1

This construction shows that, when a particle executes simple harmonic motion, if a circle be described upon the path of the particle as diameter, and a perpendicular to the path be drawn from the particle, the point in which this perpendicular meets the circle will travel round the circle with uniform velocity. The circle so described is called the *auxiliary circle.*

6. The greatest velocity of the vibrating particle will be at C, and will be equal to the velocity of the revolving point.[1] In the extreme positions A and A′, the velocity of the vibrating particle vanishes The whole motion naturally divides itself into four parts, corresponding to the four quadrants in the figure, and these four parts are reversed copies of each other.

The distance C A or C A′ from either of the extreme positions of the particle to the central position is called the *amplitude* of the vibration. It is the same as the radius of the auxiliary circle.

The *period* of the vibration is the time of moving from A to A′ and back, and is the time of a complete revolution in the auxiliary circle.

7. Let the force acting on the vibrating particle be such as to produce an acceleration μx when the distance from the centre is x. The factor μ will evidently be constant, since, from our definition of simple harmonic motion, the force is proportional to x. Now it is proved, in treatises on dynamics, that the acceleration of P is

$$\left(\frac{2\pi}{T}\right)^2 \text{C P},$$

(T denoting the period), and is directed along P C. Its component along X C is

$$\left(\frac{2\pi}{T}\right)^2 \text{C X},$$

[1] The velocity of P, and the two components of this velocity parallel to A A′ and B B′, are perpendicular to the sides of the triangle C Y P, which may therefore be taken as the triangle of velocities. Hence C Y represents the velocity of X.

and, since P and X have identical motions in this direction, this is also the acceleration of X. That is, we have

$$\mu x = \left(\frac{2\pi}{T}\right)^2 x,$$

whence

$$\mu = \left(\frac{2\pi}{T}\right)^2, \qquad T = \frac{2\pi}{\sqrt{\mu}}. \qquad (1)$$

The *period* therefore depends only on μ and is *independent of the amplitude.* In other words, the vibrating particle will make the same number of vibrations in a given time whether its excursions be large or small. This equality of period for all amplitudes is called *isochronism.* It is a general law that the small vibrations of any elastic body are isochronous; and the physical cause of this isochronism is found in the fact that the elastic resistance is proportional to the displacement. This latter fact is known as Hooke's law. It was experimentally discovered by Hooke in the case of the extension of elastic strings, and was expressed by him in the formula *Ut tensio sic vis* (As the extension, so is the force).

8. It is evident that the motion of the point Y in Fig. 1 is similar to that of X, and that the motion of P is the resultant of the two If P moves round the circle in the direction A B A′ B′, then Y will come to its extreme position B a quarter of a period later than X comes to A. Hence we have the following proposition:—

Two equal simple harmonic motions, at right angles to each other, differing in phase by a quarter of a period, compound into uniform circular motion.

9. If θ denote the angle which C P has swept out since coinciding with C A, and a the radius of the circle, we have

$$x = a \cos \theta, \qquad y = a \sin \theta.$$

Hence, simple harmonic motion may be defined as motion in which the displacement from the mean position is proportional to the sine or cosine of an angle which varies as the time.

10. If θ is measured from any fixed radius C E (Fig. 2), we shall have

$$x = a \cos (\theta - \epsilon), \qquad y = a \sin (\theta - \epsilon),$$

where ϵ denotes the angle described in moving from C E to C A; either of these formulæ may, therefore, be employed as the general expression for simple harmonic motion, it being always understood that θ is directly proportional to the time. If t denote the time occupied in describing the angle θ, and T the period, we have

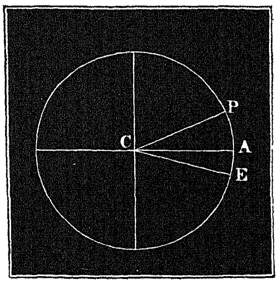

FIG 2

$$\frac{\theta}{2\pi} = \frac{t}{T}, \tag{2}$$

since θ is described in time t, and 2π in time T.

In accordance with the usage of the best modern authorities, we shall adopt as the standard formula

$$x = a \cos (\theta - \epsilon), \tag{3}$$

where ϵ, or its equivalent in time $\frac{\epsilon}{2\pi}$ T is called the *epoch*, $\theta - \epsilon$, or its equivalent in time, the *phase*, and a (as already stated) the *amplitude*.

Substituting for θ its value from (2), the formula becomes

$$x = a \cos\left(\frac{2\pi t}{T} - \epsilon\right). \qquad (4)$$

Hence the equation

$$x = a \cos(n t - \epsilon) \qquad (5)$$

denotes simple harmonic motion, whose period T is determined by

$$\frac{2\pi}{T} = n, \text{ or } T = \frac{2\pi}{n}. \qquad (6)$$

11. From the analogy of (5), which is the general equation of simple harmonic motion, one variable y is said to be a simple harmonic function of another x, when the connection between them can be expressed by the equation

$$y = a \cos(n x - \epsilon). \qquad (7)$$

12. The curve of which (7) is the equation, is called the *simple harmonic curve*. It is the curve which is traced, by a point executing simple harmonic vibrations, upon a sheet of paper travelling uniformly in a direction at right angles to the line of vibration.

For if the paper travels with velocity V in the direction of the negative axis of x, the tracing point will travel relatively to the paper in the direction of the positive axis

of x with the same velocity, so that if x denote the distance travelled in time t we shall have

$$x = \mathrm{V}\,t, \text{ or } t = \frac{x}{\mathrm{V}}.$$

But from equation (4), with y in place of x, we have

$$y = a \cos\left(\frac{2\pi t}{\mathrm{T}} - \epsilon\right),$$

that is,

$$y = a \cos\left(\frac{2\pi x}{\mathrm{VT}} - \epsilon\right).$$

Now V T evidently denotes the distance that the paper advances in the period of one complete vibration. Call this distance λ; then we have

$$y = a \cos\left(\frac{2\pi x}{\lambda} - \epsilon\right). \tag{8}$$

Comparing this with (7), we see that

$$n = \frac{2\pi}{\lambda}, \text{ or } \lambda \ \ \frac{2\pi}{n}. \tag{9}$$

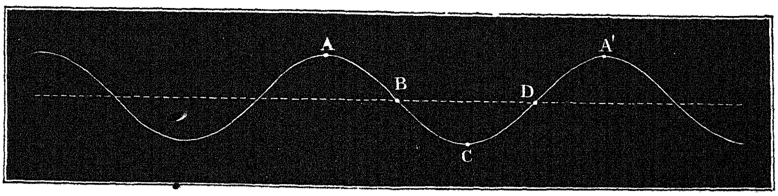

Fig. 3.

λ is called the *wave-length*. If we increase x by λ, we increase $\frac{2\pi x}{\lambda}$ by 2π, and y will be unaltered. The curve is represented in Fig. 3. It evidently consists of

a number of similar and similarly placed portions, each having a projection λ on the axis of x. The portion A A′ from one summit to the next can be divided into four parts, A B, B C, C D, D A′, which are equal and similar, but reversed in position.

13. Such curves can be traced by causing a piece of smoked glass to move rapidly past a tuning-fork which has a light style attached to one of its prongs. They can be obtained on a larger scale by attaching a pen (consisting of a glass tube drawn out to a fine point) to the lower end of a pendulum vibrating in a small arc, and causing it to write upon a sheet of paper which is drawn by clockwork in a direction perpendicular to the plane of vibration.

14. To show that the vibrations of a pendulum follow the simple harmonic law, it will suffice to consider the case of the simple pendulum, that is, of a heavy particle suspended by a weightless string, and vibrating in one plane.

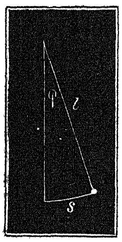

FIG. 4.

Let l (Fig. 4) be the length of the string, and ϕ the angle which it makes with the vertical at any moment. Then the acceleration of the heavy particle, being the tangential component of gravity, is $g \sin \phi$, and if s denote the distance from the lowest point, measured along the arc, we have $\phi = \frac{s}{l}$. Hence the acceleration is $g \sin \frac{s}{l}$, which, when the arc is small, may be identified with

$g\frac{s}{l}$, and is directly proportional to s. The motion of the particle, therefore, agrees with our definition of simple harmonic motion (§ 4), except that its path, instead of being straight, is slightly curved. Since the constant factor μ of § 7 has now the value $\frac{g}{l}$, we have for the periodic time (by equation 1)

$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\frac{l}{g}}. \tag{10}$$

What is commonly called the 'time of vibration' of a pendulum, is the time of swinging from one extreme position to the other, and is half the periodic time.

15. The cycloidal pendulum is an arrangement in which a heavy particle is made to oscillate, not in a circular arc, as in the case above discussed, but in a curve which fulfils the condition

$$s=k\sin\phi, \tag{11}$$

ϕ denoting the inclination of the tangent at any point to the horizon, s the distance measured along the curve from this point to the lowest, and k a constant. The acceleration of the heavy particle is

$$g\sin\phi = g\frac{s}{k},$$

which is rigorously proportional to s; and, therefore, for all vibrations, whether small or large, the periodic time is the same, its value being $2\pi\sqrt{\frac{k}{g}}$.

CHAPTER II.

GENERAL THEORY OF COMPOSITION OF MOTIONS OF TRANSLATION.

16. DEFINITION.—*If* A, B *and* C *are any three bodies, the motion of* A *relative to* C *is called the resultant of the motion of* A *relative to* B *and the motion of* B *relative to* C.

If C is regarded as at rest, the motion of A will be called the resultant of the motion of A relative to B and the motion of B.

In the present treatise we shall only have to discuss the motions of points, and we shall regard two points as having the same motion if their motions are equal and parallel; in other words, all the points of a rigid body which has a motion of translation will be regarded as having the same motion.

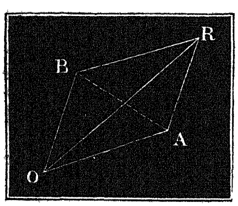

FIG. 5

17. CONSTRUCTION FOR COMPOSITION OF MOTIONS.—If A and B are any two points whose motion is to be compounded, we may take any fixed point O (Fig. 5) and construct the parallelogram of which O A, O B are two sides. If O R be the diagonal of this parallelogram, the motion of

R is the required resultant. For since A R is constantly equal and parallel to O B, the motion of R with respect to A is the same as the motion of B about the fixed point O, and the motion of R is by definition the resultant of this motion and the motion of A

If we choose different positions for O, the paths obtained for R will differ in position, but will be equal and similar, and may be regarded as the paths of different points of a rigid body which has a motion of translation.

It is not necessary to suppose the paths of A and B to lie in the plane of the paper. The construction is applicable to the movements of any two points in space.

18. THE MOTION OF THE MIDDLE POINT OF THE LINE JOINING ANY TWO POINTS A B IS HALF THE RESULTANT OF THEIR MOTIONS.

This is obvious from the figure in the preceding section, for since the diagonals of a parallelogram bisect each other, the middle point of A B is the middle point of O R, and its motion about the fixed point O is similar to that of R, but on half the scale.

19. Let x_1 x_2 x_3 be the distances of three points A B C from a fixed plane. Then, since $x_1 - x_3$ is constantly equal to the sum of $x_1 - x_2$ and $x_2 - x_3$, the motion of A relative to C resolved in a direction normal to the plane is the sum of the motions of A relative to B and of B relative to C, similarly resolved. Hence, whenever one motion is the resultant of two others, in the sense of the definition at the head of this chapter, its component in any direction

must be the sum of their components in the same direction. Conversely, this property may be taken as the definition of the resultant of two (or any number of) motions.

20. If $x_1\, y_1\, z_1$ be the distances of a point P_1 from three fixed planes at right angles to each other, with similar notation for the distances of other points $P_2\, P_3 \ldots P_n$ from the same planes, the point whose distances are

$$x = \frac{1}{n}(x_1 + x_2 + \ldots + x_n),$$

$$y = \frac{1}{n}(y_1 + y_2 + \ldots + y_n),$$

$$z = \frac{1}{n}(z_1 + z_2 + \ldots + z_n),$$

is called the *centre of mean position* of the n points, and is identical with the centre of gravity of equal masses at the n points. Its motion resolved normally to any one of the three planes is obviously $\frac{1}{n}$ of the sum of the motions of the n points similarly resolved. Hence the motion of the centre of mean position, if magnified n times, is the resultant of the motions of the n points. This proposition reduces to that of § 18 when there are only two points.

The motion of the centre of mean position may with propriety be called the *arithmetical mean* of the motions of the n given points.

21. If the line joining two points A B be unequally divided in a constant ratio, the motion of the point of

section may be called a 'mean with unequal weights.' Let the point of section be called G, and let A G be to G B as b to a, so that G is the centre of gravity of a weight a at A and a weight b at B. Then if a and b are integers, it follows from

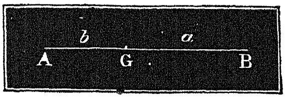

FIG 6

§ 20 that a such motions as that of A and b such motions as that of B would have for their resultant the motion of G magnified $a+b$ times. Whether a and b be integers or not, $a+b$ times the motion of G will be the resultant of a times the motion of A and b times the motion of B.

22. These principles can be illustrated by the pantagraph, an instrument used by engravers for reducing

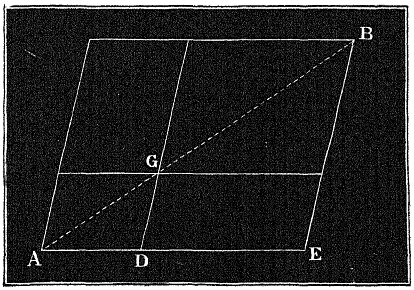

FIG 7

drawings. It may be regarded as consisting of a jointed parallelogram A B, divided by two bars parallel to its sides into four smaller jointed parallelograms, two of which, A G, G B, are similar to the whole parallelogram and are therefore about the same diagonal. The intersection G of the two cross-bars thus divides the diagonal A B in a constant ratio.

In the ordinary use of the instrument A is fixed, and a pen at G draws a reduced copy of the curve traced by a style at B, the scale of the copy being to that of the original as A G to A B, or as A D to A E, a ratio which the operator has the power to adjust at pleasure.

If A and B are simultaneously moved, G will have a motion which is a mean of their motions, and if the instrument is set for reducing one-half (in other words, if D be the middle point of A E) the motion of G will be the resultant of the motions of A and B reduced one-half. This application of the pantagraph is, we believe, new.[1]

[1] We have described the pantagraph in the manner which is simplest from a theoretical point of view. Its actual construction for the purposes of the engraver is as shown in the annexed figure. The tracing point to be

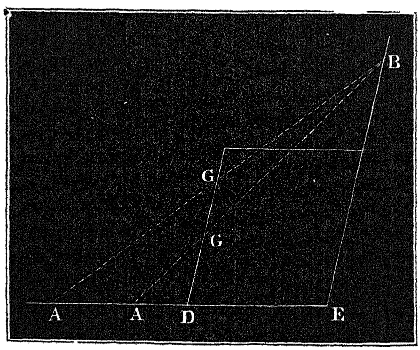

FIG. 8.

carried over the original is at a definite point B in one of the arms. The pencil or pen is at G, which is not a definite point but depends on the scale of reduction required, the bar D G being graduated for this purpose; and D A is graduated to correspond with it in such a way that when G and A are at similarly marked divisions, B, G and A will be in one straight line. A is pivoted to a heavy weight to prevent it from moving, and there are castors at the ends and corners of the frame, to roll over the paper.

If G is fixed, the motion of B will always be opposite in direction to that of A, and if A and G be simultaneously moved in given ways, it is obvious from the last sentence of § 21 that b times the motion of B will be the resultant of $a+b$ times the motion of G, and a times the reversed motion of A.

23. The arithmetical mean of the motions of three points A, B, C can be found by employing one pantagraph to give, by the intersection G of its cross-bars, the arithmetical mean of the motions of A and B; and employing a second pantagraph, with one corner jointed to G and the opposite corner to C, to give a mean in which G has double the weight of C.

To obtain the arithmetical mean of the motions of four points, we may employ one pantagraph to give the mean of the motions of two of them, a second to give the mean of the motions of the other two, and a third to give the mean of these two means.

It is thus always possible to obtain by a combination of $n-1$ pantagraphs the arithmetical mean of the motions of n points. The resultant of the motions of the n points will be this mean magnified n times.

CHAPTER III.

COMPOSITION OF VIBRATIONS OF THE SAME PERIOD.

24. Two uniform circular motions of the same period and in the same direction compound into a single uniform circular motion.

For if A and B (Fig. 9) revolve with the same period and therefore with the same angular velocity round O, the angle between the revolving radii O A, O B will be constant, and the parallelogram O A C B will revolve as a rigid figure round O. The uniform circular motion of C is the resultant of the two given motions.

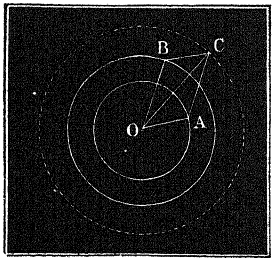

FIG. 9,

25. Two S.H. motions[1] of the same period in the same straight line compound into a single S.H. motion of the same period.

For the two components are the projections (upon the given line) of two uniform circular motions of the same period; and the resultant will be the projection of the resultant of these two circular motions. It will there-

[1] Here and elsewhere we use the initial letters S.H. as an abbreviation for 'Simple Harmonic.'

fore (by the preceding section) be the projection of uniform circular motion of the same period as either component.

26. Two uniform circular motions of the same period in opposite directions compound into S.H. motion if their radii are equal.

For if A and B (Fig. 10) revolve round the same circle with equal and opposite angular velocities, they will meet at both ends of one fixed diameter. Their motions

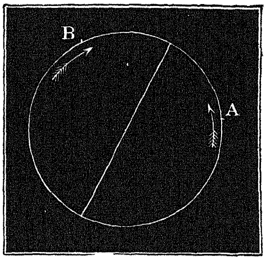

FIG. 10.

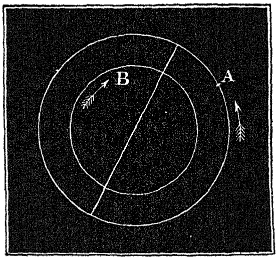

FIG. 11.

parallel to this diameter will be the same, and their motions perpendicular to it equal and opposite. Their resultant will therefore be S.H. motion along this diameter, with amplitude double the radius of either circle.

27. When the two circular motions in opposite directions have unequal radii (Fig. 11), there will still be one line passing through the common centre of the two circles such that the two revolving points A and B will cross it simultaneously twice in each revolution. Let a and b be the radii, a being the greater. Then the resultant motion parallel to this line will be the sum of two S.H. motions of

the same epoch, and will be a single S.H. motion of amplitude $a+b$. The resultant motion in the perpendicular direction will be the difference of two S.H. motions of the same epoch, and will be S.H. motion of amplitude $a-b$. The final resultant will therefore be obtained by compounding two S.H. motions of amplitudes $a+b$ and $a-b$ in perpendicular directions, with their phases so related that an extreme displacement in the one is simultaneous with a mean position in the other.

If we compare this resultant with uniform motion in a circle of radius $a+b$, we see that it agrees with it as regards one component, and that the other component has the constant ratio $\frac{a-b}{a+b}$ to that in the circle. The path will therefore be an ellipse whose major and minor semi-axes are $a+b$ and $a-b$. The resultant motion in this case is called *elliptic harmonic motion.*

28. In general any two S.H. motions of the same period compound into elliptic harmonic motion.

For let $2a$ and $2b$ be their amplitudes. Then, by § 26, the first of the two S.H. motions can be resolved into two opposite uniform circular motions of radius a, and the second into two of radius b. We have thus four uniform circular motions of the same period, two of them (of radii a and b) being in one direction, and the other two (also of radii a and b) in the opposite direction. Compounding (by § 24) those which have the same direction, the four uniform circular motions are reduced

to two in opposite directions; and these (by § 27) compound into elliptic harmonic motion.

Simple harmonic motion and uniform circular motion are extreme cases of elliptic harmonic motion.

29. The projection of S.H. motion upon any straight line is S.H. motion.

For let O A = O A′ be the amplitude of the original motion, and B, B′ the projections of A, A′ upon any line through O. The acceleration of the original tracing point P is μ. P O, which can be resolved into μ. P Q, and

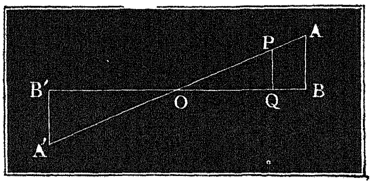

FIG 12

μ. Q O, the latter being the acceleration of Q the projection of P upon B B′. Q has therefore simple harmonic motion of the same period and epoch as P. The amplitude of the motion of Q will be O B, which is the original amplitude multiplied by the cosine of the angle between the original and the projected motion.

30 The projection of elliptic harmonic motion upon any straight line is S.H. motion. For the elliptic motion can be resolved into S.H. components along the principal axes of the ellipse; each of these will project into S.H. motion on the given line, and the resultant of these two projections will itself be a S.H. motion.

The projection of elliptic harmonic motion upon any plane is elliptic harmonic motion. For its projection upon two lines at right angles to each other in the given plane will be simple harmonic, and their resultant will (by § 28) be elliptic harmonic motion.

31. The acceleration in elliptic harmonic motion is always directed towards the centre of the ellipse, and proportional to the distance from the centre.

For it is the resultant of the two accelerations μ. X O, μ. Y O (Fig 13), along the principal axes (X and Y being

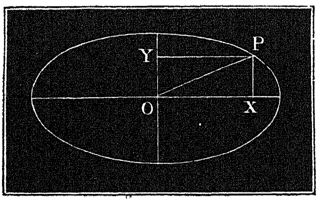

FIG 13

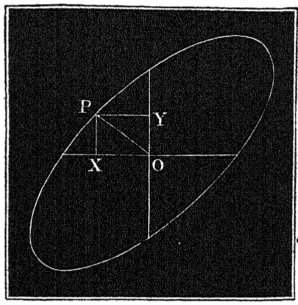

FIG. 14

the projections of P upon these axes), and these evidently compound into μ. P O.

Conversely, if a point P move in a plane curve round a point O with an acceleration always represented by μ. P O, the path will be an ellipse of which O is the centre, and will be harmonically described. For if X and Y (Fig. 14) are the projections of P on two rectangular axes through O, the accelerations of X and Y will be μ. X O, μ. Y O, so that the motions of X and Y will be simple har-

monic; and the motion of P, being their resultant, will (by § 28) be elliptic harmonic motion.

32. The resultant of any number of S.H. motions along any lines whatever, if all have the same period, is elliptic harmonic motion. For the given motions can be projected upon three rectangular axes, and each of the projections will be simple harmonic. Those which are along the same axis will compound into one S.H. motion. The required resultant will therefore be obtained by compounding three S.H. motions of the same period along three lines at right angles, and we may regard O as the central point of each of the three motions, so that, O X, O Y, O Z being the three displacements from the origin O, the accelerations will be μ. X O, μ. Y O, μ. Z O. The resultant of these will be μ. P O, P being the point whose projections are X, Y, Z. But the motion of P is the resultant motion which we are seeking. Let a plane be drawn through the tangent to the path of P at a given moment, and also through O. The whole path of P will lie in this plane, and will be the same as that of a free particle attracted towards O with a force varying as the distance. From the second part of the preceding section it follows that the motion will be elliptic harmonic.

33. We shall now investigate the amplitude of the S.H. motion which results from the composition of two S.H. motions in the same line.

Let A and B (Fig. 15) be the two points which travel uniformly round the auxiliary circles of the two compo-

nents, C the point which travels uniformly in the auxiliary circle of the resultant. Then O A, O B, O C, are the three amplitudes, and O C, being the diagonal of a parallelogram of which O A, O B are the sides, may have any value intermediate between their sum and difference, according to the magnitude of the angle A O B, which is the difference of phase (or difference of epoch) of the two components. When this angle is zero they have the same phase, and the resultant amplitude is the sum of the given amplitudes. When it is 180° they have opposite phases, and the resultant amplitude is the difference of the given amplitudes. In this case if the given amplitudes are equal the two components destroy each other, and the resultant is absolute rest.

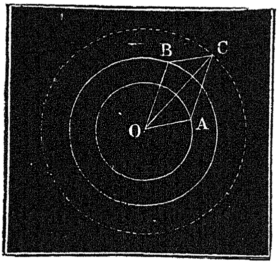

FIG. 15

The formula for the square of the amplitude of the resultant is evidently

$$\mathrm{O\,C}^2 = \mathrm{O\,A}^2 + \mathrm{O\,B}^2 + 2\,\mathrm{O\,A}.\,\mathrm{O\,B}\cos \mathrm{A\,O\,B},$$

A O B being the difference of phase of the two components, and O A, O B their amplitudes.

34. If the two component S.H. motions have not rigorously the same period, as hitherto supposed, the angular velocities of A and B in the two auxiliary circles will not be rigorously equal, and the angle A O B will change at a constant rate. We shall suppose this rate to

be slow in comparison with the angular velocities themselves, so that the angle A O B will only undergo a small change in each revolution. Then the path of C in one revolution will be nearly circular, and its velocity in this path sensibly uniform, so that the projection of its motion upon a given line will be very approximately simple harmonic; but the radius of the circle will gradually alter in successive revolutions, taking all values intermediate between the sum and difference of O A, O B. Remembering that the radius of the circle is the amplitude of the resultant, we see that the resultant of two S.H. motions of slightly unequal periods (both having the same line of motion) may be described as a S.H. motion with amplitude varying between the sum and the difference of the two given amplitudes.

35. The variation of the square of the amplitude is simple harmonic. For if we drop a perpendicular C D on O A or O A produced, we have

$$\mathrm{OC}^2 = \mathrm{OA}^2 + \mathrm{AC}^2 \pm 2\,\mathrm{OA}.\,\mathrm{AD},$$

where all the quantities on the second side are constant except A D. The variation of OC^2 is therefore the variation of 2 O A. A D, and is proportional to A D. But the motion of C relative to O A is uniform circular motion round A, and the motion of D along the line O A is therefore simple harmonic motion with A as central point.

It follows that the mean value of the term $\pm 2\,\mathrm{OA}.\,\mathrm{AD}$ is zero; and therefore the mean value of OC^2 is $\mathrm{OA}^2+\mathrm{OB}^2$,

or the mean value of the square of the resultant amplitude is the sum of the squares of the component amplitudes.

36. These principles explain the throbbing character of the sound which is produced by the combination of two sounds differing slightly in pitch. The drum of the ear vibrating under their joint influence performs vibrations whose amplitudes vary from the sum to the difference of the amplitudes due to the two separate sounds. If the separate effects of the two sounds were exactly equal and were simple harmonic, there would be momentary silence at the instant when the phases became opposite. Two 'stopped' organ-pipes mounted side by side on the same wind-chest, and tuned as nearly as possible to unison, will often maintain this opposition of phase (and almost complete extinction of sound) for a considerable time.

The alternations of loudness produced by the cause here explained are called *beats*. Each beat indicates that one of the two sources has gained a complete vibration upon the other; and hence, if the number of vibrations made by one source is known, the number made by the other can be found by adding or subtracting the number of beats.

37. They also explain the phenomena of spring and neap tides.

Speaking broadly, the variation of tidal level at a given place is the sum of two S.H. variations, one depending on the moon and the other on the sun, the

former having the larger amplitude and a rather longer period. When the phases of these two S.H. variations concur, we have spring tides with amplitude equal to the sum of the lunar and solar amplitudes, and when the phases are opposite we have neap tides with amplitude equal to their difference.

COMPOSITION OF RECTANGULAR S.H. MOTIONS.

38. We have seen that the resultant of two S.H. motions of the same period and not in the same straight line is elliptic harmonic motion. We shall now investigate the form and position of the ellipse as affected by the amplitudes and epochs of the two components. We shall in the first instance, and throughout the greater part of our discussion, suppose the two components to be at right angles, this being the only case of practical importance.

Let the directions of the two components, for convenience of language, be called horizontal and vertical. Describe two concentric circles (Fig. 16) whose radii are the two amplitudes; then one component will be the horizontal motion of a point H travelling uniformly round one circle, and the other component will be the vertical motion of a point V travelling round the other with the same angular velocity. It is optional to regard the directions of revolution in the two circles as the same or opposite; we can, therefore, without loss of generality,

suppose them to be the same. The angle H O V between the revolving radii will then be constant.

If we draw two tangents at the extremities of the horizontal diameter of the circle in which H moves, these will be the limits of the horizontal motion of the resultant; and in like manner two tangents at the extremities of the vertical diameter of the circle described by V will be the

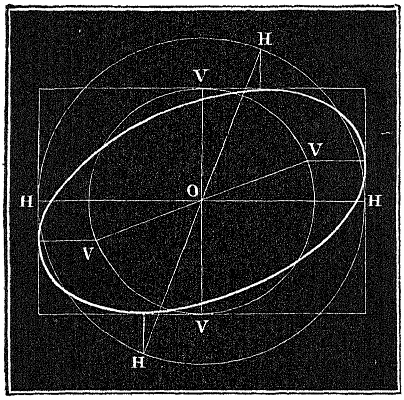

FIG 16

limits of vertical motion. Hence the ellipse will be inscribed in the rectangle formed by these four tangents.

To determine the points of contact of the ellipse with the vertical sides of the rectangle, we must consider where V will be when H is at the extremities of the horizontal diameter. V will evidently be at the extremities of a diameter inclined to the horizontal at the given angle H O V. Hence we must draw this diameter of V's circle, and from its extremities draw horizontal lines one to the right and the other to the left to meet the vertical sides

of the rectangle. The points in which they meet the vertical sides will be the points of contact of the ellipse, and will evidently be at the distance O V sin H O V from the middle points of the vertical sides. In like manner the points of contact with the horizontal sides will be at the distance O H sin V O H from the middle points of these sides (V O H being measured in the opposite direction of revolution to H O V)

When the angle H O V is zero, the points of contact will be the middle points of the sides of the rectangle

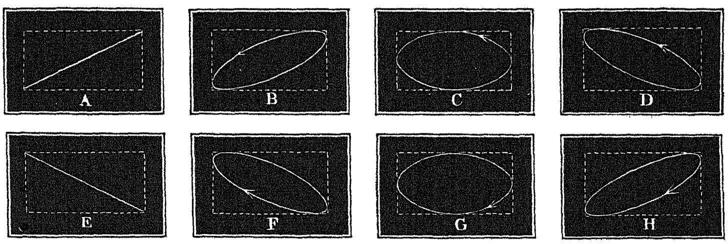

FIG 17

(Fig. 17, C), and the ellipse will be traced in the same direction of revolution as the two circles.

When H O V is a right angle, O V being 90° in advance of O H, the points of contact will (with the usual conventions as to sign) be at the top right hand and lower left hand corners of the rectangle, and the ellipse shrinks into a straight line (Fig. 17, A), namely, the diagonal joining these corners.

When O V is rather more than a right angle in advance of O H, V will have passed its highest position, and begun

to descend before H comes to the extreme right. This shows that at the point of contact of the ellipse with the right hand side, the point which traces the ellipse is descending (Fig. 17, H). The ellipse is therefore described in the opposite direction to the two circles.

When H O V is equal to two right angles, the points of contact will again be the middle points of the sides, and the ellipse will be traced in the opposite direction to the circles (Fig. 17, G).

When O V is three right angles in advance (or one right angle in arrear) of O H, the ellipse shrinks into the diagonal joining the top left hand and lower right hand corners (Fig. 17, E), and when H O V is rather more than three right angles, the ellipse will again be described in the positive direction (Fig. 17, D).

It thus appears that the ellipse is described in the positive or the negative direction according as the angle between O H and O V is acute or obtuse; and that when this angle is a right angle, the ellipse shrinks into a straight line.

39. If the periods of the two S.H. components are slightly unequal, the angle H O V will be nearly constant during the time of one revolution of H or V, but will gradually change from one revolution to the next, and will take in succession all values from 0° round to 360° or 0° again. All the forms of ellipse which we have been discussing in the preceding article will thus be traced in succession, and if a permanent mark is left by the tracing

point, it will be found to form successive lines of shading covering the whole area of the rectangle. The commencement of this process is illustrated by the last figure in Plate III., which is taken from a trace made by Donkin's Harmonograph. In this instrument, which we shall describe in a later chapter (Chapter vii. § 92), the amplitudes remain constant throughout the motion. In most (or perhaps in

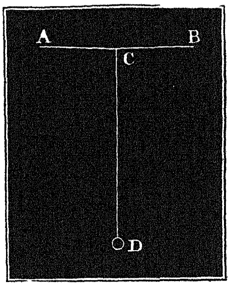

FIG. 18

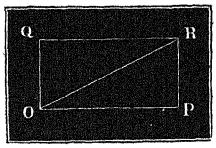

FIG 19.

all) of the other experimental illustrations of this subject, the amplitudes gradually diminish.

40 One such illustration is represented in Fig. 18. A string A C B is stretched between two fixed points A and B, in such a manner that it droops a very little in the centre; and a second string C D, carrying a weight D, is fastened to the middle point C (or to any point) of the first. If this weight is set swinging in any direction except parallel to A B or perpendicular to A B, it will trace out the series of ellipses above discussed. The following is the explanation:—

If D were set swinging parallel to A B, it would oscil-

late about C as a fixed point, since the string A C B would not move.

On the other hand, if D were set swinging perpendicular to A B, it would oscillate about the point E in which A B is cut by D C produced; for the plane of the two strings would swing from side to side, turning round A B as axis. The period of the vibration in this case would be rather longer than in the former, because the pendulum E D is longer than the pendulum C D.

If D is drawn aside in a direction oblique to A B and then let go, its motion will be compounded of two S H. motions, parallel and perpendicular respectively to A B, having these two unequal periods.

41. This result depends upon the general principle, that when the displacements of a body from its position of stable equilibrium are small enough to fulfil Hooke's law (§ 7), the force called out by any such displacement is the resultant of the two separate forces which would be called out by any two component displacements into which it can be resolved. That is to say, in the present case, if O (Fig. 19) denote the position of D when in equilibrium, and O P, O Q, two sides of a horizontal parallelogram, of which O R is the diagonal, the forces called out by displacing D along O P and along O Q will, if compounded, give a single force the same in magnitude and direction as the force called out by displacing D along O R. If we take O P and O Q parallel and perpendicular to A B, the two component forces called out will be along these

lines, and may be called μ_1 P O and μ_2 Q O. The resultant force will not be along the diagonal R O unless μ_1 is equal to μ_2; and in the present case μ_1 is greater than μ_2. We shall return to this subject in a later chapter. (Chapter vi. § 74.)

42 We append confirmatory proofs by co-ordinate geometry of some of the foregoing results.

Composition of any number of S.H. motions in the same straight line with the same period

Their equations may be written (§ 10)

$$\left.\begin{aligned} x_1 &= a_1 \cos(\theta-\varepsilon_1), \\ x_2 &= a_2 \cos(\theta-\varepsilon_2), \\ x_3 &= a_3 \cos(\theta-\varepsilon_3), \\ &\&c.\ \&c. \end{aligned}\right\} \qquad (1)$$

Since $\cos(\theta-\varepsilon)$ is $\cos\theta\cos\varepsilon+\sin\theta\sin\varepsilon$, we have

$$x_1+x_2+x_3+\&c. = \cos\theta\,(a_1\cos\varepsilon_1+a_2\cos\varepsilon_2+\&c.)$$
$$+\sin\theta\,(a_1\sin\varepsilon_1+a_2\sin\varepsilon_2+\&c.) \qquad (2)$$

Put

$$\left.\begin{aligned} a_1\cos\varepsilon_1+a_2\cos\varepsilon_2+\&c. &= \mathrm{A}\cos\mathrm{E} \\ a_1\sin\varepsilon_1+a_2\sin\varepsilon_2+\&c. &= \mathrm{A}\sin\mathrm{E} \end{aligned}\right\} \qquad (3)$$

Then we have

$$x_1+x_2+\&c = \mathrm{A}\cos\mathrm{E}\cos\theta+\mathrm{A}\sin\mathrm{E}\sin\theta \qquad (4)$$
$$= \mathrm{A}\cos(\theta-\mathrm{E}).$$

Hence the resultant is S.H. motion of amplitude A and epoch E. The value of A^2 will be obtained from equations (3) by squaring and adding; and then tan E will be found by dividing the second equation by the first.

It may be remarked that equations (3) are of the same form as the equations for finding the resultant of a set of forces acting at a point in one plane.

43. Compositions of two S.H. motions of the same period at right angles, differing by a quarter-period in epoch.

If the equation of one of them be

$$x=a\cos\theta,$$

that of the other will be

$$y=b\cos\left(\theta\pm\frac{\pi}{2}\right) = \mp b\sin\theta.$$

Hence

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=\cos^2\theta+\sin^2\theta=1, \qquad (5)$$

the equation to an ellipse of semi-axes a b.

44. Composition of any two S.H. motions of the same period, at right angles.

Their equations may be written—

$$x=a\cos\theta,$$

$$y=b\cos(\theta-\delta)=b(\cos\theta\cos\delta+\sin\theta\sin\delta).$$

But $\cos\theta=\frac{x}{a}$, and $\sin\theta=\frac{1}{a}\sqrt{(a^2-x^2)}$.

Hence we have

$$y=\frac{b}{a}\left\{x\cos\delta+\sqrt{(a^2-x^2)}\sin\delta\right\},$$

or

$$\left(y-\frac{bx}{a}\cos\delta\right)^2=\frac{b^2(a^2-x^2)}{a^2}\sin^2\delta. \qquad (7)$$

This is an equation of the second degree, and as we

know that the resultant motion must be in some closed curve, the locus is, in general, an ellipse.

The following are particular cases :—

$$\delta=0 \quad \text{gives } y=\frac{b\,x}{a}, \quad \text{a straight line.}$$

$$\delta=\frac{\pi}{2} \quad ,, \quad y^2=\frac{b^2}{a^2}\,(a^2-x^2), \text{ as in § 43.}$$

$$\delta=\pi \quad ,, \quad y=-\frac{b\,x}{a}, \text{ another straight line.}$$

$$\delta=\frac{3\,\pi}{2} \quad ,, \quad y^2=\frac{b^2}{a^2}\,(a^2-x^2), \text{ same as } \delta=\frac{\pi}{2}.$$

Describe a rectangle whose sides are the double amplitudes of the two given components. Then all the ellipses obtained by giving different values to δ can be inscribed in this rectangle, since the extreme values of x are always $\pm a$, and the extreme values of y are $\pm b$.

It will be found upon examination that the direction in which the moving point travels round the ellipse depends upon the value of δ, and is reversed as δ passes through the values 0 and π. We shall examine the two cases $\delta=\frac{\pi}{2}$, $\delta=\frac{3\pi}{2}$, for which we have obtained in last article the same equation $y^2=\frac{b^2}{a^2}\,(a^2-x^2)$.

When $\delta=\frac{\pi}{2}$,

$$\theta=0 \text{ gives } x=a,\, y=0,$$

$$\theta=\frac{\pi}{2} \quad ,, \quad x=0,\, y=b.$$

Hence in one quarter period the motion is from the point $x=a$, $y=0$ to the point $x=0$, $y=b$. The direction of revolution is therefore from the positive axis of x to the positive axis of y.

$$\text{When } \delta=\frac{3\pi}{2},$$

$$\theta=0 \text{ gives } x=a,\ y=0,$$

$$\theta=\frac{\pi}{2} \quad ,, \quad x=0,\ y=-b.$$

Hence the direction of revolution is from the positive axis of x to the negative axis of y; which is opposite to the direction of revolution in the preceding case.

45. Next, let us suppose the periods of the two mutually perpendicular components to be only approximately equal. Then the resultant motion at any moment will be approximately one of the ellipses represented by equation (7), but δ will gradually change, and thus, instead of the motion repeating itself in a fixed ellipse, it will approximate in succession to *all* the ellipses which equation (7) can be made to represent by giving every possible value to δ. The first figure in Plate III. shows the trace left, by a point describing these approximate ellipses, upon a sheet of paper travelling uniformly past it.

If δ is increasing, the x vibrations are gaining upon the y vibrations. For the phase of the x vibrations is θ, and the phase of the y vibrations is $\theta-\delta$; but if δ increases with θ (and at a much slower rate) the increment of $\theta-\delta$ in any time is less than the increment of θ; that is to say, the phase of the y vibrations increases more

slowly than the phase of the x vibrations. The opposite will be the case if δ is decreasing.

45.* To investigate the resultant of two S.H. motions along lines inclined at an angle other than a right angle, we have only to suppose the axes of co-ordinates in the preceding sections 43-45 to be oblique. The algebraic work is unchanged, and the interpretation of the results presents no difficulty.

CHAPTER IV.

WAVES.

46. Let $a a_1$, $b b_1$, $c c_1$, $d d_1$, &c. (Fig. 20), be equal parallel and equi-distant straight lines, such that their extremities a, b, c, d, &c, are in one straight line, and their other extremities therefore also in a straight line. Let there be a number of particles, one in each of the lines $a a_1$, $b b_1$, &c., executing simple harmonic vibrations

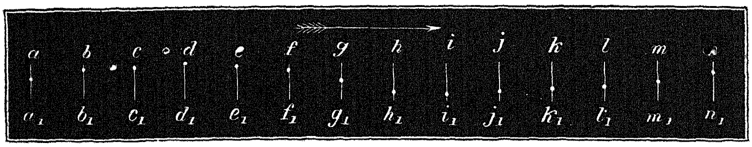

Fig 20

in them, so that a and a_1, are the extreme positions of the first particle, b and b_1, of the second particle, and so on. Let the periods of their vibrations be all equal, but let the second particle be a little later in its phases than the first, the third a little later than the second, and so on, the differences of phase from particle to particle being all equal. Then we shall have a simple harmonic undulation—in other words, a series of simple harmonic waves—traversing the particles.

In the figure, the first particle is supposed to be at the middle of its descent from a to a_1, and the seventh particle to be at the middle of its ascent from g_1 to g. The wave is travelling from left to right.

47. Let the common difference of phase from particle to particle be $\frac{1}{p}$ of the common period, p being a large integer, then the difference of phase between the first particle and the $p+1$th, or between the second particle and the $p+2$th, or between the rth particle and the $p+r$th will amount to one period. But a difference of phase of one period means identity of phase. Hence the phase of the first particle at any moment is the same as that of the $p+1$th, also of the $2p+1$th, and the $3p+1$th, the phase of the second particle is the same as that of the $p+2$th, the $2p+2$th, the $3p+2$th, &c., and in general the phase of the rth particle is the same as that of the $p+r$th, the $2p+r$th, the $3p+r$th, and so on.

The first p particles will form one complete wave, the next p particles another complete wave, which at any given moment will be precisely similar to the first; the next p particles will form another similar wave, and so on. *Any* p successive particles will form one complete wave, and if we include $p+1$ particles, the phases of the first and last will be identical. The distance from the first particle to the $p+1$th (in other words, p times the common distance $a\,b$ or $b\,c$ or $c\,d$) is called the length of the waves, or the *wave-length*, and is usually denoted by

the Greek letter λ. Stated in general terms, the wave-length of a system of equal and similar waves is the distance from any particle to the next particle that is in the same phase, the distance being measured not along the curved outline of the wave, but along the straight line which is the direction of propagation of the wave (the line $a\, b\, c\, d$, or the line $a_1\, b_1\, c_1\, d_1$, in the figure).

48. Let us suppose, for convenience of language, that the lines of vibration $a\, a_1$, $b\, b_1$, &c., are vertical, and the direction of wave-propagation horizontal. Then a particle which is in its highest position is at the crest of a wave, and a particle in its lowest position is at the trough of a wave; and these particles will again assume these positions when the present wave is replaced by its successor; in other words, when the distance that the waves have advanced amounts to one wave-length. Thus we arrive at the following important relation between the period of vibration of the particles, the wave-length, and the velocity of propagation of the waves.

The distance that the waves travel in one period is one wave-length; or if v denote the velocity of wave-propagation.

$$\lambda = v\,\mathrm{T}. \tag{8}$$

If the period T be $\frac{1}{n}$ of the unit of time, so that each particle makes n vibrations in the unit of time, the above equation may be written in the form

$$v = n\,\lambda, \tag{9}$$

which expresses that the distance travelled by the waves in the unit of time is n wave-lengths.

49. We shall now investigate the equation of the curve formed by the particles at any time t, taking as axis of x the line which passes through the middle points of the paths of the particles, and taking any point in this line as origin. The ordinate y will be the displacement of any particle from its mean position.

The displacement of the particle whose mean position is at the origin may be represented by the equation

$$y = a \cos \theta,$$

where a is the amplitude, and θ is $\frac{t}{\mathrm{T}} 2\pi$ (see § 10), or $\frac{v\,t}{\lambda} 2\pi$, by equation (8).

As we pass from one particle to another, in the direction in which the waves travel, x will increase, and the phase will diminish at such a rate that when x increases by λ the phase diminishes by 2π; the equation for the displacement of any particle is therefore

$$y = a \cos\left(\theta - \frac{x}{\lambda} 2\pi\right),$$

that is,

$$y = a \cos\left(\frac{v\,t - x}{\lambda} 2\pi\right). \qquad (10)$$

which is the required expression. It shows that the curve formed by the particles at any time is the harmonic curve (§ 12), for equation (10) agrees with equation (8) when t is constant. Dividing numerator and denominator

by v, and remembering that $\frac{\lambda}{v} = T$, we have

$$y = a \cos \left(\frac{t - \frac{x}{v}}{T} 2\pi \right), \qquad (11)$$

which is therefore another form of the equation to the wave-curve at any time t.

When the lines of motion of the particles are, as in the present case, perpendicular to the line of propagation of the waves, the vibrations are said to be *transverse*; but the equations which we have obtained are equally applicable when the vibrations are parallel to the line of propagation, or have any intermediate direction, if we regard y as denoting the displacement of the particle whose mean co-ordinate is x.

50. We shall now examine the relation between the velocity of a particle and the position of the particle on the waves.

We know that the velocity in simple harmonic motion is greatest at the central point and diminishes to nothing at the extremities. Hence a particle at the crest of a wave, as A (Fig. 3), or at the trough, as C, is for the instant at rest, and the maximum velocities are at B and D, the points midway between crest and trough. If the waves in the figure are travelling from left to right, B is rising with maximum velocity, and D is falling with maximum velocity. All particles between A and C are rising, and all particles between C and the next crest A′ are falling.

If we consider two points very near together on a wave, it is evident that they rise or fall through the difference of their heights during the time in which the waves advance through their horizontal distance. Hence, *the difference of their heights, divided by their horizontal distance, is equal to the velocity of either particle divided by the velocity of propagation of the waves.* That is to say, in the language of the differential calculus,

$$\frac{dy}{dx} = -\frac{\dot{y}}{v} \qquad (12)$$

where $\dot{y}$ denotes $\frac{dy}{dt}$, the negative sign being prefixed because $\frac{dy}{dx}$ is negative from A to C, where the particles are rising, or, in other words, where $\dot{y}$ is positive.

This conclusion can be verified by differentiating equation (10). For, writing that equation in the form

$$y = a \cos \theta, \text{ where } \theta = \frac{vt - x}{\lambda} 2\pi,$$

we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}, \quad \frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt},$$

therefore

$$\frac{dy}{dx} : \frac{dy}{dt} \quad \frac{d\theta}{dx} : \frac{d\theta}{dt};$$

but

$$\frac{d\theta}{dx} = -\frac{2\pi}{\lambda}, \text{ and } \frac{d\theta}{dt} = \frac{2\pi v}{\lambda},$$

hence

$$\frac{dy}{dx} : \frac{dy}{at} :: -1 : v,$$

which agrees with (12). $\frac{dy}{dx}$ is obviously the tangent of the inclination of the curve to the axis of x.

51. Next let the vibrations be *longitudinal*; that is, let the lines of motion of the particles be coincident with the direction of propagation of the waves.

The particles will then remain always in one straight line, but their alternate forward and backward movements will bring them sometimes nearer together and sometimes further apart; so that instead of crests and troughs we shall have compressions and extensions propagated along the line of particles. Equation (10) or (11) still represents the displacement of any particle from its mean position, and we shall regard y as positive when the displacement is forward (understanding by the forward direction the direction in which the waves are propagated, which is the same as the direction of the positive axis of x). Hence, where we had before a crest we shall now have a maximum forward displacement, and where we had before a trough we shall now have a maximum backward displacement.

To calculate the compressions and extensions, let x be the mean co-ordinate of one particle, that is, the co-ordinate of the particle when in its mean position, and $x+\delta x$ the mean co-ordinate of a particle a little in

advance of it. Also let y and $y+\delta y$ be their respective displacements from their mean positions. Then the distance between the two particles, instead of being the distance δx between their mean positions, is $\delta x+\delta y$. The measure of the extension is the increase of distance δy divided by the mean distance δx, or more strictly is the limit of this ratio $\frac{dy}{dx}$. When $\frac{dy}{dx}$ is positive, it indicates extension, and when negative, compression.

From the equation

$$\frac{dy}{dx}=-\frac{\dot{y}}{v}$$

we infer that particles which are moving forward (and for which therefore $\dot{y}$ is positive) are in compression, that particles which are moving backward are in extension, and that the amount of compression or extension is directly as the forward or backward velocity of the particles, being equal to the quotient of this velocity by the velocity of propagation of the waves.

It is by longitudinal vibrations that sound is propagated through air, and through all gases and liquids.

CHAPTER V.

COMPOSITION OF TWO SIMPLE HARMONIC UNDULATIONS OF EQUAL WAVE-LENGTH.

52. By calling an undulation the resultant of two others, we mean that the motion of each particle is the resultant of the motions due to these two component undulations. In general the resultant of two displacements is the diagonal of the parallelogram of which the two component displacements are the sides; but we shall for the present confine our attention to the case in which the two components are parallel. The resultant will then be their algebraic sum. In every case we shall suppose the velocity of propagation to be the same for both the component undulations.

53. First, let the direction of propagation be the same for both, and their wave-lengths equal. Since the velocities of propagation are also equal, the periods will be equal, and the difference of phase will be a definite quantity, the same for all the particles and remaining constant. Let this difference of phase be ϕ, and let the amplitudes of the two undulations be a and b. The motion of each particle is then the resultant of two simple

harmonic motions in the same straight line, with a difference of epochs ϕ. This resultant is (by § 33) a simple harmonic motion of the same period as the components, and having an amplitude

$$c = \sqrt{(a^2+b^2+2\,a\,b\cos\phi)}$$

represented by the diagonal of the parallelogram, whose sides are the two component amplitudes a and b placed at an angle ϕ.

54. If the wave lengths of the two component undulations are only approximately equal, while their velocities of propagation are still supposed to be exactly equal, the difference of epoch ϕ will have different values at different points at the same moment, or for the same particle at different moments. The amplitude of vibration of each particle will, on the principles of § 34, alternately increase and diminish, its maximum value being the sum, and its minimum value the difference of a and b.

55. These results can be verified by means of the standard equation (10) of wave motion.

Denoting for shortness $\dfrac{vt-x}{\lambda}\,2\pi$ by θ, we may write the equations of the two undulations, on the assumption that λ is the same for both,

$$y_1 = a\cos\theta,$$
$$y_2 = b\cos(\theta-\phi),$$

and the equation of the resultant undulation will be

$$y = a\cos\theta + b\cos(\theta-\phi)$$
$$= \cos\theta\,(a+b\cos\phi) + \sin\theta\,(b\sin\phi)$$

This will take the form

$$y=c\ (\cos\theta\cos\psi+\sin\theta\sin\psi)\ =c\cos(\theta-\psi),$$

if we have

$$c\cos\psi=a+b\cos\phi,$$
$$c\sin\psi=b\sin\phi.$$

These two last equations give, by squaring and adding,

$$c^2=a^2+b^2+2\,a\,b\cos\phi,$$

and by division they give

$$\tan\psi=\frac{b\sin\phi}{a+b\cos\phi}.$$

Hence c and ψ can always be determined in terms of the given quantities a, b, and ϕ, and y can thus be reduced to the form

$$y=c\cos(\theta-\psi),$$

that is

$$y=c\cos\left(\frac{vt-x}{\lambda}2\,\pi-\psi\right),$$

where c and ψ are independent both of x and t. This last equation evidently denotes a simple harmonic undulation of wave length λ, velocity v and amplitude c.

56. On the supposition of a slight difference in the wave-lengths of the two components, the circumstances are approximately represented by making ϕ vary slowly with t. Then the greatest value of c^2 will be obtained by putting $\phi=0$, and will be $a^2+b^2+2\,a\,b$ or $(a+b)^2$; and the least will be obtained by putting $\psi=\pi$, and will be $(a-b)^2$. The mean value of c^2 will (as explained in § 35) be a^2+b^2.

57. The case discussed in §§ 54–56 is that which occurs when two simple tones of nearly the same pitch are emitted from two neighbouring sources of sound. The waves from the two sources are propagated through the surrounding air with the same velocity, and those which belong to the sound of higher pitch are slightly shorter than the others.

58. Next, let the directions of propagation be opposite for the two component undulations, their wavelengths being exactly equal, and also their amplitudes.

First, suppose the vibrations to be transverse. Then the successive actions will be understood by inspection of the annexed figures 21, 22, 23, 24. The heavy curve in each figure represents a definite portion A B C D A′ of the vibrating string, and the two lighter curves above it represent the two component undulations travelling in opposite directions, as suggested by the arrows. The interval of time from each figure to the next is a quarter period.

In Fig. 21 there is a coincidence of crests at B and of troughs at D.

In Fig. 22 a crest coincides with a trough at A, C, and A′.

In Fig. 23 there is coincidence of troughs at B and of crests at D.

In Fig. 24 there is a coincidence of crest with trough at A, C, and A′. After another quarter period the state of things in Fig. 21 will recur

In each quarter period one component has travelled a quarter wave-length to the left, and the other the same distance to the right, so that their displacement relative to each other is half a wave-length.

Comparing together the four positions of the string

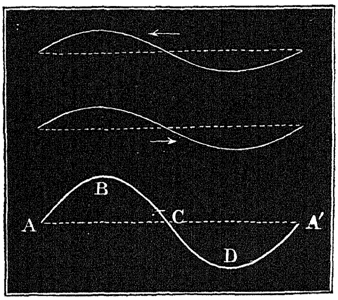

FIG 21

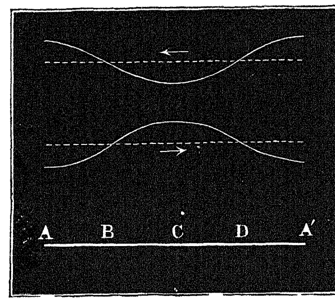

FIG 22

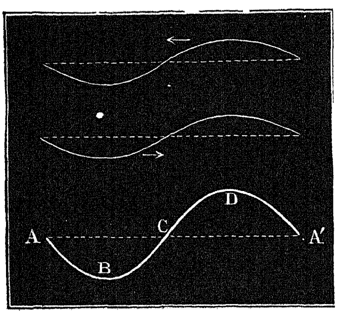

FIG 23

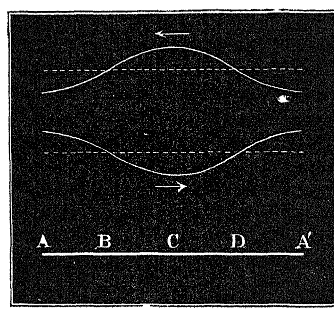

FIG 24

here represented, it is easy to understand that the points A, C, and A′ remain permanently at rest, and the points B and D midway between them undergo the largest displacements. On the other hand, if we attend to the direction of a tangent at a given point of the string, we see that this direction changes most at A, C, and A′, and

does not change at all at B and D. The points of permanent rest, A, C, and A′ are called *nodes*, and the points of maximum displacement, B and D, *antinodes*.

59. Now, passing to the case of longitudinal vibrations, consider the state of things at the moment when the maximum compressions in the two sets of waves are in coincidence, and, therefore, the maximum extensions also in coincidence At this moment the effect of one set of waves, as regards compression or extension, is at every point the same as the effect of the other set, both in kind and in degree, and the resultant effect will be double of either.

But as regards velocity the effects destroy one another; for at the places of compression the velocity due to either set is in the direction of propagation of that set, and these two directions of propagation are opposite; while at the places of extension the velocities are opposite to the directions of propagation, and, therefore, are here also opposite to each other. Hence, there is instantaneous rest all along the waves.

Next, consider the state of things at the moment when the maximum compressions of one set coincide with the maximum extensions of the other. We shall then have mutual destruction of effect as regards compression and extension, but the velocities due to the two sets will be identical, and therefore the resultant velocity will be double of either.

The time intervening between the two moments here

considered is only a quarter of a period; for since the two sets of waves are travelling opposite ways, their relative velocity is double the absolute velocity of either; and the interval of time in question is the time of travelling over half a wave-length with this double velocity, or is the time in which each set of waves advances a quarter of a wave-length.

60. After another quarter period the waves will have advanced another quarter of a wave-length, and we shall again have double compressions and extensions, with instantaneous rest everywhere; but as the waves have advanced half a wave-length since the moment first considered, the places of compression and extension are interchanged. The points which were then the places of maximum compression are now the places of maximum extension, and *vice versâ*. Hence there are certain points, situated at regular intervals of half a wave-length from each other, which are alternately the points of maximum compression and of maximum extension. At these points the compression or extension is constantly double of that due to either set of waves separately, and the velocity is constantly zero—in other words, these are places of permanent rest. They are called *nodes*.

61. When the maximum compressions of one set are in coincidence with the maximum extensions of the other, the points at which these coincidences occur are midway between the nodes. At these points (which are called *antinodes*) the velocities at the moment now

under consideration are double of the maximum velocity due to either set separately, and are in opposite directions at any two successive antinodes. The velocity at these points is constantly double of that due to either set of waves separately, and the compression due to one set is constantly destroyed by the extension due to the other set.

62. We shall now establish these results as deductions from the equations of wave motion. Opposite directions of wave propagation must be distinguished by opposite signs of v. Hence, since the equation of waves travelling in the direction of $+x$ is

$$y_1 = a \cos \frac{x - vt}{\lambda} 2\pi, \tag{13}$$

the equation of equal waves travelling in the direction of $-x$ will be

$$y_2 = a \cos \frac{x + vt}{\lambda} 2\pi. \tag{14}$$

The equation of the resultant waves is obtained by adding these, and is

$$\begin{aligned} y &= a\left\{\cos \frac{x - vt}{\lambda} 2\pi + \cos \frac{x + vt}{\lambda} 2\pi\right\} \\ &= 2a \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}. \end{aligned} \tag{15}$$

Hence we have

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}, \tag{16}$$

$$\frac{dy}{dt} = -\frac{4\pi va}{\lambda} \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}. \tag{17}$$

Remembering that y denotes displacement, $\frac{dy}{dx}$ extension, and $\frac{dy}{dt}$ velocity, we can draw the following inferences from these equations.

(A) At the points for which $\sin\frac{2\pi x}{\lambda}=0$, and therefore $\cos\frac{2\pi x}{\lambda}=\pm 1$, there will be no extension or compression, but the displacements and velocities will be greater than at any other points. These are the antinodes. To find them we must put

$\frac{2\pi x}{\lambda}=m\pi$, where m is any integer, and we have

$x=m\frac{\lambda}{2}$

(B) At the points for which $\cos\frac{2\pi x}{\lambda}=0$, and therefore $\sin\frac{2\pi x}{\lambda}=\pm 1$, the displacements and velocities are constantly zero, but the compressions and extensions are greater than at any other points. These are the nodes. To find them we must put

$$\frac{2\pi x}{\lambda}=m\pi+\frac{\pi}{2}, \text{ whence } x=m\frac{\lambda}{2}+\frac{\lambda}{4}.$$

(C) At the times for which $\sin\frac{2\pi vt}{\lambda}=0$, and therefore $\cos\frac{2\pi vt}{\lambda}=\pm 1$, the velocity is everywhere zero,

and the displacements and compressions or extensions are everywhere greater than at other times. To find these times we must put

$$\frac{2\pi\, vt}{\lambda} = m\pi\,;\ \text{or, since } \lambda = v\text{T},$$

$$\frac{2\pi\, t}{\text{T}} = m\pi, \text{ whence } t = m\,\frac{\text{T}}{2}.$$

(D) At the times for which $\cos \frac{2\pi\, vt}{\lambda} = 0$, and therefore $\sin \frac{2\pi\, vt}{\lambda} = \pm 1$, there is no displacement nor extension or compression anywhere, but the velocity is everywhere greater than at other times. To find these times we must put

$$\frac{2\pi\, vt}{\lambda} = \frac{2\pi\, t}{\text{T}} = m\pi + \frac{\pi}{2}, \text{ whence } t = m\,\frac{\text{T}}{2} + \frac{\text{T}}{4}.$$

The origin from which x is measured in the equations which we have been using is evidently an antinode.

63. Passing now to the case in which the vibrations are transverse, equations (13) to (17) will still hold, y still denoting displacement and $\frac{dy}{dt}$ velocity; but $\frac{dy}{dx}$, instead of denoting extension, will denote the tangent of the inclination of the wave to the axis of x. The nodes, or places of permanent rest, will be at the same points as before, as will likewise the antinodes or places of maximum departure from rest. The tangent to the wave

remains constantly parallel to the axis of x at the antinodes, and departs furthest from parallelism at the nodes.

64. The undulation represented by equation (15), which may be written

$$y = 2a \cos \frac{2\pi\, x}{\lambda} \cos \frac{2\pi\, t}{T},$$

is called a *stationary* simple harmonic undulation. A cord vibrating in the manner represented by this equation does not present the appearance of waves running along it, but swings from side to side, while certain points in it, namely its nodes, remain fixed.

The investigation which we have given shows that a stationary simple harmonic undulation can be resolved into two S.H. undulations, each of half its amplitude, travelling in opposite directions with equal velocities.

A vibrating musical string behaves as if it were a portion of a string of indefinite length, along which two equal sets of waves travel in opposite directions.

65. We shall now discuss a few cases of the composition of S.H. undulations lying in two planes perpendicular to each other.

Let z denote distance measured along the line of mean positions of the particles, x and y the component displacements of a particle from its mean position.

To find the resultant of two stationary S.H. undulations, lying in perpendicular planes and having the same nodes, we must combine the two equations

$$x = 2a \cos \frac{2\pi z}{\lambda} \cos \frac{2\pi t}{T},$$

$$y = 2b \cos \frac{2\pi z}{\lambda} \cos\left(\frac{2\pi t}{T} - \delta\right);$$

or, writing

A for $2a \cos \frac{2\pi z}{\lambda}$, B for $2b \cos \frac{2\pi z}{\lambda}$, and θ for $\frac{2\pi t}{T}$,

$$x = A \cos \theta, \quad y = B \cos (\theta - \delta);$$

and § 44 shows that the paths of the particles will in general be ellipses (the straight line and the circle being particular cases). A and B vary from particle to particle in proportion to $\cos \frac{2\pi z}{\lambda}$. Their ratio is the same for all the particles, and δ is also the same; hence all the ellipses will be similar and similarly placed. Also, since the ratio of x to y is the same for all the particles, they all lie at a given moment in one plane, which revolves round the axis of z in the line T.

66. To find the resultant of two S.H. undulations lying in perpendicular planes and travelling in the same direction, both having the same wave-length, we must combine the equations

$$x = a \cos \frac{z - vt}{\lambda} 2\pi,$$

$$y = b \cos \left(\frac{z - vt}{\lambda} 2\pi - \delta\right);$$

or, writing θ for $\frac{z - vt}{\lambda} 2\pi$,

$$x = a \cos \theta, \; y = b \cos (\theta - \delta).$$

Hence in this case also the motion of each particle is in general elliptic. Since a and b are constants, all the ellipses will be equal and will have the same projection on the plane of x, y. The value of θ is proportional to $z-vt$; hence different particles will have the same value of θ at different times connected by the relation

Difference of $z = v \times$ interval of time.

In the particular case in which $\delta = \pm \frac{\pi}{2}$, and $a=b$, the particles will describe circles with uniform velocity, and the whole motion will be that of a corkscrew rotated on its axis without longitudinal motion. The corkscrew (or *helix*) will be right-handed or left-handed according to the sign of δ.

67. If the wave-lengths of the two components are only approximately equal, the ellipses described by the particles will gradually change in the manner described in § 45. This result is very important in connection with what is called *elliptic polarization* in optics.

CHAPTER VI.

COMPOSITION OF TWO S.H. MOTIONS OF DIFFERENT PERIODS.

68. Thus far we have confined our attention to the composition of motions of the same or nearly the same period. We now proceed to some cases not thus limited.

First, let the two S.H. motions to be compounded be at right angles to each other, and have periods in the ratio of 1 to 2.

Let T be the period for the x component, and T$'$ the period for the y component; let θ stand for $\frac{2\pi t}{\mathrm{T}}$ and θ for $\frac{2\pi t}{\mathrm{T}'}$. Then we have, by § 10,

$$x=a\cos(\theta-\epsilon),$$
$$y=b\cos(\theta'-\epsilon').$$

But $\frac{\theta}{\theta'}=\frac{\mathrm{T}'}{\mathrm{T}}$; hence if T is double of T$'$, θ' is double of θ.

Also, by reckoning time from an instant when x has its maximum value, we make $\epsilon=0$, and, putting δ for the difference of epoch $\epsilon'-\epsilon$, we shall have

$$x=a\cos\theta$$
$$y=b\cos(2\theta-\delta). \qquad (12)$$

If we eliminate θ from these equations we obtain an

equation of the fourth degree, which is not of much interest, but in two particular cases the second half of the curve coincides with the first—the moving point retracing its course—and the equation reduces to the second degree. One of these is the case of $\delta = 0$, which gives

$$\frac{y}{b} = \cos 2\theta = 2\cos^2\theta - 1 = \frac{2x^2}{a^2} - 1, \text{ or}$$

$$x^2 = \frac{a^2}{2b}(y + b),$$

the equation of a parabola, whose vertex is at a distance b from the central point of the vibrations, and focus at the distance $\frac{a^2}{8b}$ from the vertex.

The other is the case of $\delta = \pi$, which gives

$$\frac{y}{b} = -\cos 2\theta, \text{ and } x^2 = \frac{a^2}{2b}(-y + b),$$

denoting the same parabola inverted, its concavity being now turned towards the negative instead of the positive axis of y.

Of the other curves, the most interesting is the symmetrical figure of 8 which corresponds to $\delta = \frac{\pi}{2}$ and $\delta = -\frac{\pi}{2}$, the path being the same for both these cases, but traced in opposite directions.

The first line of Plate II. shows these and some of the intermediate forms.

69. Whenever the two periods are commensurable, the

curve described returns into itself, and the period of its description is their least common multiple. Let the ratio of T to T′ (reduced to its lowest terms) be that of $m : n$ Then the equations will be

$$x=a\cos n\theta \qquad (13)$$
$$y=b\cos(m\theta-\delta).$$

The curve will be inscribable in a rectangle whose sides are $2a$ and $2b$, since the extreme values of x are $\pm a$, and the extreme values of y are $\pm b$. Each of the two extreme values of x is attained n times, and of y, m times, in the complete period.

When δ is 0 or π, the tracing point will go twice over its path—once forward and once backward; for in the former case we have

$$\frac{x}{a}=\cos n\theta,\ \frac{y}{b}=\cos m\theta;$$

and $-\theta$ gives the same values both of x and y as $+\theta$; that is to say, the tracing point will be in the same spot at equal distances of time before and after the beginning of each complete period; this beginning being fixed by the equations at a time when both x and y have their maximum positive values.

70. In like manner, when $\delta=\pi$, the equations

$$\frac{x}{a}=\cos n\theta,\ \frac{y}{b}=-\cos m\theta$$

show that the tracer will be in the same place at equal intervals of time before and after the moment when x

and y have their extreme values, one positive and the other negative.

By giving δ the value $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ we obtain a curve possessing special features of symmetry. In the former case we have

$$x_1 = a \cos n\theta,$$
$$y_1 = b \sin m\theta,$$

and in the latter

$$x_2 = a \cos n\theta,$$
$$y_2 = -b \sin m\theta.$$

Every point on the locus of $x_1 y_1$ is also on the locus of $x_2 y_2$, and corresponds to a value of θ the same in magnitude but opposite in sign. Hence the two curves are the same, but are traced in opposite directions.

71. The curves obtained by compounding two simple harmonic motions are often exceedingly beautiful. and the movements of the pen in tracing them, when it travels at a convenient speed for the eye to follow, are graceful in the extreme. Several examples are represented in Plate II., the specimens selected being for the most part either the symmetrical curve just described, or the curve which terminates abruptly at the ends, the tracer returning upon itself, as described in § 69. The ratio of the two periods is indicated in each case. (See also § 94.)

72. The following is the geometrical construction for tracing these curves by points.

Describe two concentric circles of radii equal to the

amplitudes of the two components. Let H be the point which by its motion round one circle gives the horizontal displacement, and V the point which by its motion round the other gives the vertical displacement. Then, having selected the starting points H_0 and V_0 so that V_0 is $90°-\delta$ in advance of H_0, find the intersection of a vertical through H_0 with a horizontal through V_0. This will be one point on the curve. Set off in the forward direction a succes-

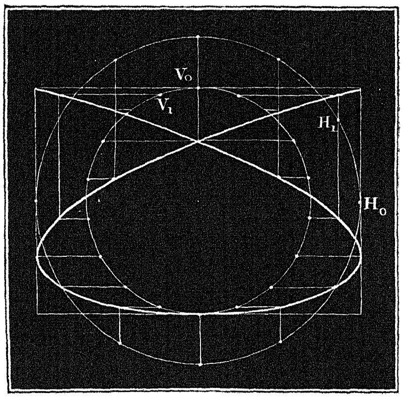

FIG. 25

sion of equal arcs $H_0 H_1$, $H_1 H_2$, &c., of any convenient magnitude on the one circle, and on the other equal arcs $V_0 V_1$, $V_1 V_2$, &c., such that the latter are to the former (when expressed in degrees) as the period of the horizontal to the period of the vertical vibrations (or as m to n in equation 13, if x is horizontal and y vertical). Find the intersections of a vertical through H_1 with a horizontal through V_1, a vertical through H_2 with a horizontal through V_2, and so on, until the curve obtained begins to return

into itself, which will in general be after travelling n times round one circle and m times round the other.[1]

Fig. 25 illustrates the application of this method to the curve whose equations are

$$x = a \cos 3\theta, \; y = b \cos 2\theta.$$

V_0 the starting point for y is one right angle in advance of H_0 the starting point for x, and the curve begins to be retraced backwards after $\frac{3}{2}$ revolutions of H and one revolution of V.

73. When the ratio of the two periods is approximately but not exactly that of two small whole numbers m and n, the curve described will approximate in succession to each of the curves obtained by giving different values to δ. If δ is increasing (of course at a very slow rate compared with $m\theta$), the increase of $m\theta - \delta$ is less than the increment of $m\theta$, and thus the y vibrations, which depend upon $\cos(m\theta - \delta)$, are slower than they ought to be in comparison with the x vibrations.

74. To describe any of these curves by means of the simple apparatus described in § 40 (which is called *Blackburn's Pendulum*), the lengths of the strings must be so regulated that if E is the point where A B would be cut by

[1] If the number of vibrations made in the unit of time be called their *frequency* (so that period and frequency are reciprocals), the arcs set off in the two auxiliary circles are to be directly as the frequencies In the vibrations represented by the equation

$$y = b \cos(m\theta - \delta)$$

the period is $\frac{1}{m}$ of the time in which θ increases from zero to 2π, hence the frequency is directly as m

C D produced, E D is to C D as m^2 to n^2; inasmuch as the period of vibration of a pendulum is as the square root of the length.

75. They may also be obtained by means of the vibrations of a straight rod, fixed at one end and free at the other. If the rod offers the same resistance to bending in all longitudinal planes, the free end will describe either an ellipse, a circle, or a straight line, according to the manner in which it is started; but if it is not equally stiff for all directions of bending, there will be two directions at right angles to each other, in one of which the resistance is a maximum and in the other a minimum, and the vibrations actually executed will be compounded of two simple harmonic vibrations in these directions.

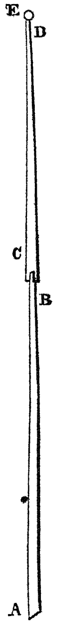

FIG 26

If the section of the rod be a rectangle of sides a and b, or an ellipse whose length and breadth are a and b, the resistances to equal displacements of the free end parallel to a and b respectively will be as a^2 to b^2, and the periods of vibration in these directions will be as b to a.

Rods constructed for showing the composition of these vibrations are called *Kaleidophones*, and the best form (on account of the great variety of curves that it can show) is the *double-spring Kaleidophone*, represented in Fig. 26.

76. Two long narrow and flat pieces of steel, A B, C D,

are joined in one straight line, with their planes at right angles. The lower piece A B is fixed in a vice or similar clamp at any point in its length that may be desired, so that the portion of A B above the clamp, and the whole of C D, are free to vibrate. A bright bead E is firmly attached to the top of C D, and serves as a moving point of light, which, by the persistence of impressions in the eye, leaves a luminous track as the observer looks down upon it.

When the upper end of C D is drawn aside and then let go, it does not return directly towards the position of equilibrium unless the displacement be in one of the two planes A B or C D. If the lower piece is clamped very near its upper end, there will be great resistance to displacement in the plane C D, and the vibrations in this plane will accordingly be much quicker than those in the perpendicular plane. If the clamp is near the lower end A, there will be little resistance to displacement in the plane C D, and the vibrations in this plane will be slower than those in the perpendicular plane. There is one definite position of the clamp which will make the times of vibration in the two planes equal; and as we move the clamp either upwards or downwards from this, the times of vibration become more and more unequal. The ratios 1 : 2, 2 : 3, 3 : 4, &c., can thus be obtained each in two ways.

77. A third method is furnished by *Tisley's Harmonograph* or compound pendulum apparatus, the working of

which will be understood from Fig. 27. Two pendulums oscillate in perpendicular planes, their axes of suspension being at some distance below their upper ends. Two light rods attached to these upper ends by ball and socket joints are jointed at their further ends to the penholder, and thus enable each pendulum to push and pull the pen in a direction parallel or nearly parallel to its own plane of vibration. The pen is a vertical glass tube drawn out to a fine point below, and draws the curves upon a card laid beneath it. As the amplitudes of the vibrations of the pendulums gradually diminish, the curves become gradually smaller, and very beautiful effects of shading are thus obtained. The first two figures in the last line of Plate III. are examples. The ratio of the periods is indicated in each case.

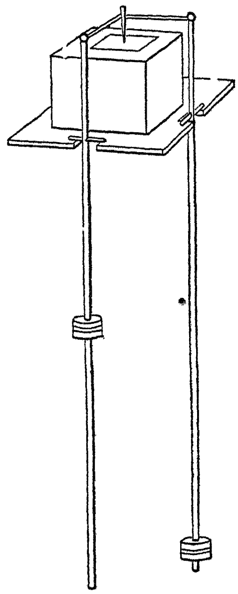

FIG 27

In another form of the apparatus, one of the pendulums carries at its top a table, on which the card is to be laid, and the other pendulum, by means of an arm, carries the pen. The curves are thus produced by the movement of the card in one direction, combined with the movement of the pen in the perpendicular direction.

78. A fourth method, due to Lissajous, is represented in Fig. 28. Two large tuning forks vibrate, one in a

horizontal and the other in a vertical plane, each having a small mirror clamped to one prong near the end, the plane of the mirror being parallel to the flat of the prong, and a beam of light from a small but powerful source is reflected from one mirror to the other and thence to a screen, a lens being interposed between the source and the first mirror, at a proper distance for bringing the beam to a focus on the screen. When one of the forks vibrates, the ray reflected from it swings to-and-fro in the plane of vibration through a small angle, and if the other fork is stationary the image on the screen moves to-and-fro in a straight line, presenting (by the persistence of impressions) the appearance of a line of light on the

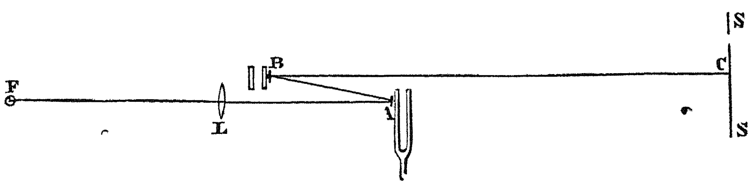

FIG. 28

screen. When the other fork alone vibrates, the appearance is that of a line of light perpendicular to the former; and when both vibrate, these two rectangular movements are compounded, producing luminous curves of the forms discussed in the preceding sections. As the curves first became popularly known by this experiment, they are often called Lissajous' figures.

Fig. 28 represents a section of the apparatus made by a plane containing the whole course of the beam of light,

which we may conveniently suppose to be horizontal F is the source of light, consisting of a lamp flame shining through a hole in one side of a metallic chimney, L the lens (its conjugate focal distances being F L and L A + A B + B C), A the mirror attached to the horizontal fork, B the mirror attached to the vertical fork (the two prongs of which are seen in section), C the spot of light on the screen S S. The movement of A causes C to move along the line S S, and the movement of B causes C to move perpendicular to the plane of the diagram.

79. Another mode of obtaining them by means of tuning forks is to attach the object glass of a compound microscope to a prong of one tuning fork, the eyepiece and tube being attached to a fixed support, and to observe through the microscope a small bright object, such as a grain of starch, attached to a prong of the other fork. The focal length of the lens should be two or three inches, and the vibrations of the two forks must be in parallel planes but in perpendicular directions. If the two forks are exactly in unison, or if the interval between their notes be an absolutely true musical interval, there will be no change in the curve observed, after the forks have been started and left to themselves, except such as results from the gradual diminution of the amplitudes; but if the interval be ever so little untrue, the gradual change of form will be readily detected. This optical test of the exactness of the tuning of a fork is more delicate than any that is furnished by the sense of hearing.

80. If a tracing point travels on the surface of a circular cylinder, with a motion compounded of uniform rotation round the cylinder and S.H. motion parallel to its axis, the projection of its motion upon a plane parallel to the axis is the resultant of two rectangular S.H. motions; for the component parallel to the axis is unchanged, and the rotational component projects into S.H. motion perpendicular to the axis. The relative epoch δ in equations (13) will depend on the position of the plane of projection with respect to the figure traced on the cylinder; and if the cylinder carrying the figure with it be turned on its axis through any angle while the plane of projection remains fixed, δ will be changed by an amount equal to this angle. The appearance actually seen in either form of Lissajous' experiment is very suggestive of a cylindrical curve thus rotating uniformly on its axis; the apparent rotation being uniform because the change of δ takes place at a uniform rate.

When the periods of the two components are equal, we have seen that the figure corresponding to a certain value of δ is a straight line. Hence, by the above construction, the curve on the cylinder for equality of periods is a plane section of the cylinder, and is therefore an ellipse.

All the Lissajous' figures obtained by giving different values to δ in the equations

$$x = a \cos n\,\theta, \qquad y = b \cos (m\,\theta - \delta), \qquad (14)$$

are projections of the curve obtained by winding round

a cylinder of circumference $\frac{2\pi a}{n}$ a S.H. curve of amplitude b and wave-length $\frac{2\pi a}{m}$. The number of wave-lengths required will be m, and they will go n times round the cylinder. The axis of this cylinder is parallel to the axis of y. If we take the axis of our cylinder parallel to the axis of x, we must interchange a with b and m with n in this description.

81. If a point vibrates with a motion compounded of two simple harmonic motions parallel to the axis of y, and leaves a trace upon paper which moves uniformly in the direction of the negative axis of x, the equation to the trace will be

$$y = a \cos mx + b \cos (nx - \delta). \qquad (15)$$

The first term will not be altered if we increase x by $\frac{2\pi}{m}$, which may accordingly be called the wave-length of this term. In like manner the wave-length of the second term will be $\frac{2\pi}{n}$, and the least common multiple of these two wave-lengths will be the wave-length of the resultant curve.

When m and n are equal, the resultant curve will, by § 33, be a simple harmonic curve of the same wave-length as each of its components, and of amplitude having any value between the sum and the difference of a and b, according to the value of δ. It will be the sum when $\delta = 0$, and the difference when $\delta = \pi$.

82. If δ increases or diminishes at a rate which is very slow compared with the increase of nx and mx, we shall have a succession of different curves, each nearly identical with that which would be given by a constant value of δ, this constant value being different for successive curves The case of two S.H. vibrations whose periods are approximately in the ratio of two small integers can be reduced to this. For example, the equation

$$y=a \cos 30\, x+b \cos 19\, x,$$

can be written

$$y=a \cos 30\, x+ b \cos (20\, x-x),$$

which is of the form of equation (15) with $m:n$ as $3:2$, and $\delta=x$.

When the ratio of $m:n$ is approximately unity, we have what may be called the curve of beats. For example, the equation

$$y=a \cos 55\, x+b \cos 54\, x,$$

consists of a series of approximately S.H. curves, their amplitudes varying from the sum to the difference of a and b. This is obvious from § 34.

Plate I. contains six specimens of the curve

$$y=a \cos mx+b \cos nx,$$

with m and n approximately in the ratio of two small integers, and $a=b$. The value of a or b is the same for all the curves. The approximate ratios are marked on the left hand, and the accurate ratios on the right.

CHAPTER VII.

MECHANICAL ILLUSTRATIONS OF SIMPLE HARMONIC MOTION.

83. In this chapter we propose to describe instances of simple harmonic motion not depending on vibrations of pendulums, or on forces of elasticity, but on arrangements for the transmission and transformation of motion, and especially on arrangements for transforming circular into rectilinear motion.

84. Uniform circular motion can be converted into simple harmonic motion by the arrangement shown in Fig. 29. A B is a piece which by means of the guides G G

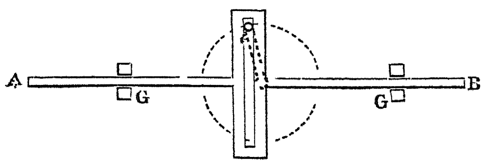

Fig 29

is constrained to travel in a straight line; and in one part of it there is a slot at right angles to the line of travel. A crank revolving uniformly in a circle (both indicated by dotted lines) passes through this slot, which it fits so accurately that there is no shake, and at the same time

no pinching The slot and the piece A B in which it is cut will evidently be compelled to take a simple harmonic motion.

This plan involves a large amount of friction ; and it would be difficult to preserve a good fit, as the parts would wear loose. Instead of allowing the crank to rub against the slot, it is better, as suggested by the Rev. F. Bashforth, in 1845, to make the crank work in a circular hole in the centre of a sliding-piece which travels in the slot. Fig. 30, which is copied from Mr. Bashforth's

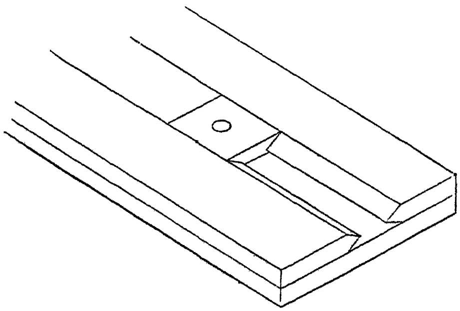

FIG. 30.

drawing, shows the slot and sliding-piece with the hole in its centre.

85. A motion approximately simple harmonic can be obtained by the arrangement shown in Fig. 31. A B is a

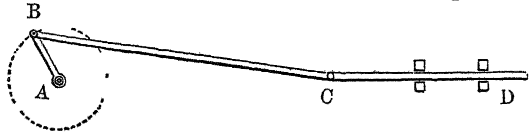

FIG. 31

crank revolving uniformly round the centre A. C D is a rod guided to move in a straight line passing through A,

and B C is a connecting-rod jointed at B and C to the other two pieces. C D will have a reciprocating motion, which will be more nearly simple harmonic as the ratio of the length of the connecting-rod to that of the crank is greater.[1] The arrangement here described is met with in many of the commonest forms of steam-engine, C D being the piston-rod and A B the crank, which it drives with a nearly uniform velocity of rotation; the uniformity being maintained by means of a fly-wheel, or in the locomotive by the inertia of the engine and train. A similar arrangement is also generally employed for working the slide-valves, the crank A B being usually replaced by an eccentric mounted on the axle of the fly-wheel. If B be the centre of this eccentric and A the fixed point round which B revolves, the motion of the connecting-rod B C and of the valve-rod C D will be precisely the same as if A B were a crank.

86. In steam-engines which admit of being reversed, the apparatus for reversing consists usually of a combination of two eccentrics, each having its own connecting-rod for giving an approximately simple harmonic motion to the slide-valves. The principle of its action is illustrated by Fig. 32, where A and B are the two eccentrics, their centres revolving in one and the same circle (the

[1] If the projection of B C upon the straight line A C D can be regarded as of constant length, the motion of C is the same as the motion of the projection of B, and is therefore simple harmonic In order that the motion of C may be *sensibly* simple harmonic, the difference between the greatest and least projections of B C (the former being B C itself) must be negligible in comparison with the amplitude A B

dotted circle in the figure) round a fixed centre. The line joining the centres of the two eccentrics is always a diameter of this circle, so that when one is in the extreme position to the right the other is in the extreme position to the left. C D is the valve-rod, which is constrained to travel in what is very nearly a straight line, passing through the centre of the dotted circle. The ends, E F, of the two connecting-rods are joined by a piece in which

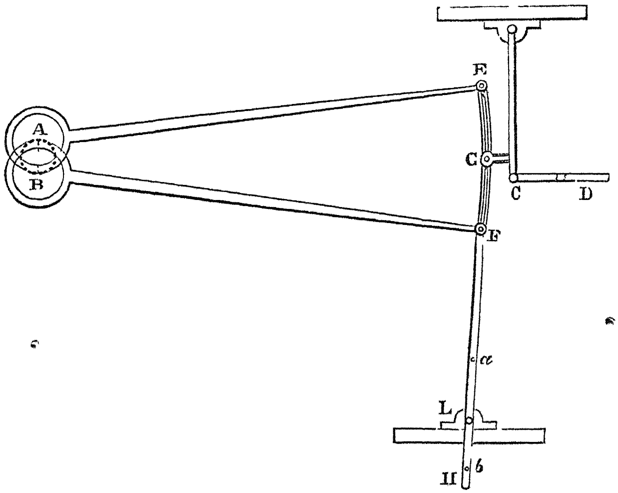

FIG. 32.

a slot is cut for the purpose of receiving a button G, which moves with the valve-rod. In the figure the button is represented as midway along the slot. In this position, the opposite motions of the points E F combine to leave the button and valve-rod nearly at rest. This accordingly is the position for stopping the engine.

By means of the bar F H, which contains three holes

for fixing it by a pin L, the slot can be made to travel along the button. When the hole a is brought down to L, the upper end of the slot will be brought down to the button, and the movement of the button will be governed by the eccentric A. This is the position for driving the engine in one direction—say forwards. Then, to reverse the engine, the hole b is brought to L, and the lower end of the slot is thus brought to the button, which will now be driven by the eccentric B instead of by A, and the

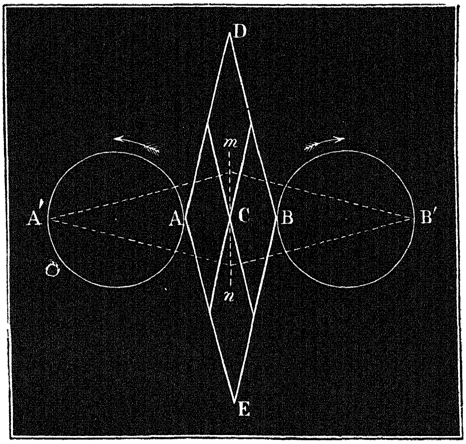

FIG. 33.

phases of motion for the valve-rod (and consequently fo the piston) will be reversed.

87. New method of obtaining S.H. motion.

A rigorous simple harmonic motion can be obtained without the friction of guides by employing a pantagraph (as described in § 22) to give the arithmetical mean of two equal and opposite uniform circular motions (see § 26), as

illustrated by Fig. 33. A D B E is a jointed rhombus, with two cross-bars jointed to the middle points of its sides. A pair of opposite corners, A B, are to be carried with equal and constant velocities in opposite directions round two equal circles in the same plane, so that the line joining A B is always parallel to the line of centres. Then the other diagonal D E will always bisect the line of centres at right angles, and the point C, in which the cross-bars intersect, being identical with the intersection of the two diagonals, will be the projection of A or B upon the fixed line D E. The point C will therefore have simple harmonic motion, of amplitude equal to the radius of either circle. The motion of C could be magnified or diminished in any required ratio by means of a second pantagraph with one corner fixed, and either the intersection of its cross-bars or the opposite corner attached to C. The dotted line *m n* in the figure represents the path of C, and the dotted parallelogram represents another position of the rhombus.

We have for simplicity supposed the parallelogram A D B E to be a rhombus and the diagonal A B to be parallel to the line of centres; but it is evident from §§ 22 and 26 that these restrictions are not necessary. The generalised construction will be as follows:—

Let A D B E be any parallelogram with two cross-bars jointed to the middle points of its sides; and let a pair of opposite corners A B be carried, with equal and constant velocities in opposite directions, round two equal

circles either in the same plane or in parallel planes; then the intersection of the cross-bars will have S.H.

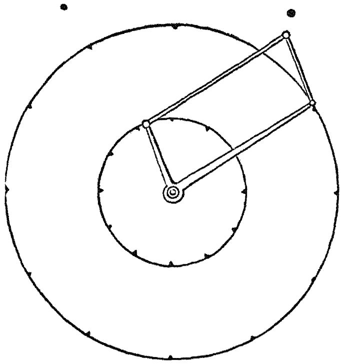

FIG 34.

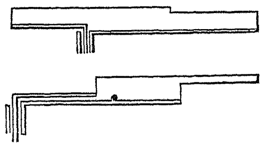

FIG 35

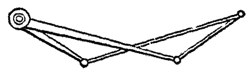

FIG 36.

motion. The simultaneous starting points may be any two points in the two circles.

88. If we attempt to compound two opposite circular motions or two circular motions with unequal angular velocities in the same direction, by means of a jointed parallelogram (Fig. 34) with one corner fixed and the two adjacent sides revolving round it, we are met by a serious practical difficulty; for though, by placing the sides of the parallelogram in different planes, we may succeed in making them clear one another at the dead points (that is, the positions in which all four are in one straight line), as shown in the sections Fig. 35, there is nothing to prevent the frame from losing the parallelogram shape in passing these points, and changing into the form shown in Fig. 36.

89. Simple harmonic motion can also be obtained by

making a circle roll uniformly inside another circle of double its diameter.

For, let A P C (Fig. 37) be the smaller circle, B its centre, A the centre of the larger circle, and E A P D one of its diameters. Then, because A is on the circumference of the smaller circle, the angle P B C at its centre is double of the angle A The arc P C is therefore equal to twice A B multiplied by the circular measure of A, that is to the arc D C. From this equality of the two arcs, it follows that P comes in contact with the larger circle at D Hence the path of P is the diameter D E. Also, since the angle A P C in a semicircle is a right angle, P is the projection of the point of contact C; and as C travels uniformly round the larger circle, P has simple harmonic motion.

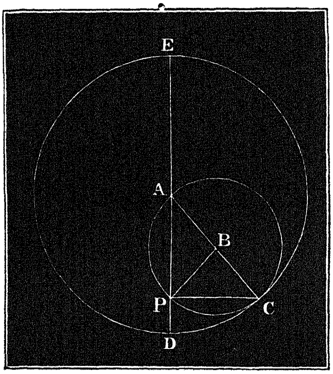

FIG 37

90. It is worthy of remark that the motion of P may be regarded as the resultant of the motion of the centre B of the smaller circle, and the motion of P relative to B.

Now B describes a circle with uniform velocity round the fixed centre A; and, since the line B P revolves uniformly, the motion of P relative to B is a circle described with uniform velocity These two circles have equa radii (the radius of the smaller circle), and they are de-

scribed in the same time (the time in which the smaller circle rolls over the whole circumference of the larger); also the directions of revolution in them are opposite, and each of them is described uniformly. Hence, by § 26, the motion of P is rectilinear and simple harmonic, with amplitude double of the radius of the smaller circle, or equal to the radius of the larger. The result above obtained is thus confirmed.

This proof further shows that the motion of any point carried by the rolling circle and not lying upon its circumference, is compounded of two uniform and unequal circular motions of the same period in opposite directions, and is therefore (by § 27) elliptic harmonic.

91. The two following methods of obtaining S H. motion are perhaps rather of theoretical than practical interest.

If the extremities of a straight line A B, of constant length, are constrained to travel along two straight lines at right angles to one another, the middle point C of A B (since the middle point of the hypotenuse of a right angles triangle is equally distant from the three corners) will describe a circle round the point of intersection O of the two fixed lines. If we call these two lines horizontal and vertical, it is evident that A's motion is double of the horizontal motion of C, and B's motion is double of the vertical motion of C; hence, if C revolve uniformly, the motions of A and B will be simple harmonic, with amplitudes double of the radius of C's circle.

To put this method in practice, we may replace the two fixed lines by two grooves, with a sliding piece tapered off at the ends travelling in each. The moving line A B will be a straight bar with its ends jointed to these sliding pieces. One end of a revolving arm O C, of length equal to half the bar, must be jointed to the middle point of the bar, and its other end must be jointed to a point whose projection on the plane of the grooves would be their intersection. As the bar A B has to sweep over a considerable space all round O, the attachments for supporting the fixed end of the revolving arm O C must be at a considerable distance.

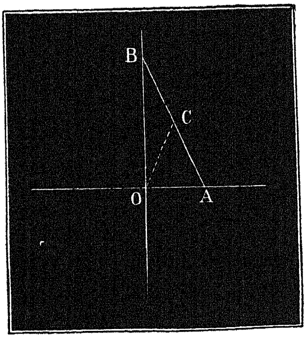

FIG 38

92. This latter drawback can be avoided by discarding one of the grooves and one half of the straight bar; the remaining half A C and the revolving arm O C jointed to it will form two sides of a variable isosceles triangle, whose base, O A, is a portion of the groove. The objection to this plan is that when the sliding piece is near the middle point of its path (that is when A is near O) the pull or

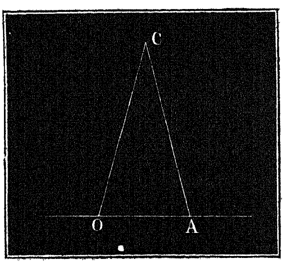

FIG. 39.

thrust along A C is nearly perpendicular to the groove, and tends to jam the sliding piece against one side of the groove. The middle point itself is a dead point, the statically applied force there being perpendicular to the required motion.

93. A machine for compounding two parallel simple harmonic motions with any given ratios of period and of amplitude, has been invented by Mr. A. E. Donkin, and is constructed by Tisley and Company.

There are two vertical axles turning in fixed positions, and carrying cranks whose lengths are made equal to the amplitudes of the two motions which are to be compounded. Toothed wheels of various sizes are provided, which can be fixed on the axles (one on each), and the numbers of their teeth determine the ratio of the periods of the two components. As their axes are fixed, a third toothed wheel with movable axis must be employed to connect them; and by this means, when one of the two fixed axles is turned by hand, the motion is transmitted to the other with the required velocity-ratio. The crank carried by the first axle gives, through the medium of a long connecting-rod, a vibratory movement to the lower end of a lever, the upper end of which moves the pen (a glass tube drawn out to a point). The paper is drawn uniformly, in a direction perpendicular to the movement of the pen, by the revolution of a roller, carrying a toothed wheel, which is driven by a train of wheelwork deriving its motion from the revolution of the

above-mentioned vertical axles. One of these, as above stated, gives a vibratory movement to the pen. The other, by means of its crank and a long connecting rod, gives a vibratory movement of translation to the frame on which the paper rests. The roller which draws the paper is also carried by this frame, and it is necessary that the toothed wheel on the roller should remain in gear with the pinion which is to drive it, in spite of this vibratory motion (which is parallel to the axes of the toothed wheel and pinion). This object is attained by making the pinion of great length (a long fluted cylinder), so that the toothed wheel can slide along it longitudinally. The speed of the paper and the lengths of the two cranks, as well as their velocity-ratio. can be regulated at pleasure, so as to give any required amplitudes and wave-lengths to the two undulations which are compounded. The figures in Plate I. are slightly reduced from curves traced by this machine. The approximate ratio of the two periods is stated on the left and their rigorous ratio on the right. The amplitudes of the two components are equal in these specimens, but the machine admits of their being varied independently.

94. By means of a bell-crank lever, the motion of the pen above described can be exchanged for a motion in the perpendicular direction, and thus vibrations at right angles to each other can be compounded. The figures in Plate II. are slightly reduced from curves thus drawn upon paper fixed to the frame, and not drawn onwards

by the roller; the change from one curve to the next of its kind being effected by unclamping the toothed wheels and turning one of them through a definite number of teeth before clamping again. The ratios of the periods are indicated in the margin. The following is a more complete account of the contents of the Plate.

In all the curves in the first line the ratio of the periods is 2 : 1, two horizontal vibrations being executed in the same time as one vertical. The first and the last figure in this line are parabolas.

In the second line, the ratio for the first four curves is 3 : 2, and for the last two curves 3 : 1, three horizontal vibrations being made in the same time as two or one vertical.

In the third line, the ratio is 5 : 3 for the first three curves, and 5 : 4 for the last three; five horizontal vibrations being made in the same time as three or four vertical.

In the fourth line, the ratio is 9 : 8 for the first three, and 10 : 9 for the last three, the number of horizontal vibrations being in each case greater by unity than the number of vertical vibrations.

All these ratios can easily be verified by inspection of the curves. For this purpose the student must count how many times he crosses over the horizontal breadth of the figure, and how many times over its height, in travelling along the curve, from one end of it to the other, if it has ends, or until it brings him back to the

point from which he started, if it be an endless curve. In the latter case it is convenient to select a starting-point as near as possible to one corner of the circumscribed rectangle.

On putting in wheels corresponding to a ratio which is approximately that of two small integers, the curves will gradually change of themselves, and will be found to cover with shading the whole surface of a certain rectangle. The commencement of this process is exhibited in the last figure of Plate III., the ratio here being approximately that of equality.

If, instead of the paper being fixed to the frame, it is slowly drawn on by the roller, the curves are somewhat distorted, but the order of succession is clearly put in evidence, and the working is much more rapid. The traces thus obtained, five specimens of which are given in Plate III., often bear a striking resemblance to letters of ordinary writing, and might be taken as the foundation of a natural alphabet of quickly-written characters. The approximate ratios are indicated on the left hand of the Plate, and the rigorous ratios on the right, the number of vertical vibrations being in each case greater than the number of horizontal. The horizontal amplitudes are equal to the vertical amplitudes. All the curves except the first are on the same scale, both as regards amplitude and the action of the roller in drawing the paper onwards. In the first curve, the amplitudes are much larger in comparison with the motion

due to the roller, and hence the intersections are more numerous.

95. A more elaborate combination of parallel simple harmonic motions is furnished by the tide-predicting machine of Sir William Thomson.

The variation of tidal level at a given port is approximately the resultant of two simple harmonic variations, their periods being respectively half a lunar day and half a solar day, and the amplitude of the former being in general rather more than double that of the latter. When the phases of the two concur we have spring tides, and when they are in opposition we have neap tides.

To obtain a closer approximation, the variation of tidal level must be regarded as the resultant of a much larger number of simple harmonic components, the periods of which are known from astronomical considerations, and are the same for all ports, while the amplitudes and the epochs of maximum for the separate components will be very different for different ports. These epochs and amplitudes for a given port can be calculated from a year's continuous record of tidal level at that port (better from several years' record), and when they have been thus ascertained, the tidal level at any future moment can be predicted from them. The tide-predicting machine is intended for making such prediction in the form of a continuous curve, whose ordinates are the heights of the tide. The principle of its working is illustrated by Fig. 40.

A number of pulleys, one for each simple harmonic component, are carried by cranks which are made to revolve by clockwork, the length of each crank being proportional to the amplitude of the corresponding component. The axles are ranged in two rows, an upper and a lower, and a flexible wire passes alternately below a lower and above an upper pulley The distances of the

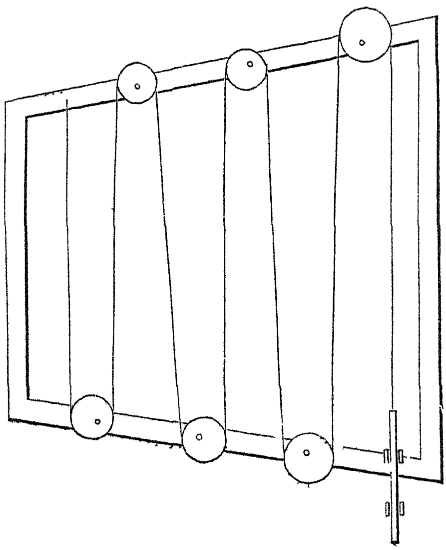

FIG 40

upper from the lower pulleys are very great compared with the lengths of the cranks, and the portions of wire which run from each pulley to the next are always nearly vertical. One end of the wire is fixed, and the other carries a heavy pen, which is guided to move only in one vertical line. The pen keeps the wire tight, and is raised and lowered by the movements of the pulleys. If only

one pulley moved, it would give the pen a simple harmonic motion, the amplitude of which would be double the length of the crank, because the height of the pulley determines the lengths of two of the free portions of wire. Hence, when all the pulleys are moving, the motion of the pen is the resultant of as many simple harmonic motions as there are pulleys.

The axles are all driven by the same clockwork, a system of toothed wheels being employed to give them approximately the correct velocity-ratios.

96. The first machine of this kind had ten pulleys, and was constructed for the Tidal Committee of the British Association. A second machine on a larger scale, with twenty pulleys (and therefore giving the resultant of twenty S. H. components) has been constructed for the Indian Government and used for computing the tides at the principal Indian ports

Its pulleys do not travel in circles like those of the machine above described, but in vertical lines on the crank-and-slot principle of § 84, so that their motions are rigorously simple harmonic.

The setting of the machine for the amplitudes and epochs of a given port occupies only a few minutes, and the tidal curve for a year can be drawn in four hours. Plate IV. is a representation, on about $\frac{1}{10}$th of the original scale, of the tidal curve traced by this machine for the nineteen days commencing midnight August 1, and ending midnight, August 20, 1881, for Beypore in

India. For the original from which it is reduced we are indebted to Mr. E. Roberts, of the Nautical Almanack Office, who has charge of the machine and its working.

This method of combining a number of S.H. motions appears to have been first invented by the Rev. F. Bashforth, who printed and circulated a lithographed description of it in 1845, from which Fig. 30, in § 84, is taken. An abstract of Mr. Bashforth's paper will be found in the 'British Association Report' for that year (Transactions of Sections, pp. 3, 4). It was reinvented by Mr. W. H. L. Russell (see 'Philosophical Magazine,' 1870), and afterwards by Mr. Beauchamp Tower, who suggested it to Sir W. Thomson when in search of some convenient method for combining such motions with a view to the graphical prediction of tides.

CHAPTER VIII.

PROPAGATION OF SONOROUS UNDULATIONS.

97. THE propagation of sound depends upon the elasticity of the medium through which the sonorous undulations are propagated. For example, when a tuning-fork vibrates in air, it gives the air a series of pushes, each of which produces a momentary increase of pressure and density in front of the advancing prongs, while a momentary decrease of density and pressure is produced behind them. As the prongs advance, first in one direction and then in the opposite, a series of compressions and extensions are produced in alternate succession. But each compressed portion tends to relieve itself by expanding into the neighbouring air, which is thus in its turn compressed, and the extended portions in like manner tend to communicate extension. Hence a series of compressions and extensions are propagated through the surrounding air, and these constitute an undulation, whose period is the same as that of the vibrations of the tuning-fork. The velocity of propagation is independent of the period, and depends only on the elasticity and density of the air, being (as we shall prove in Chapter X.) directly

as the square root of the coefficient of elasticity, and inversely as the square root of the density.

98. The *coefficient of elasticity* is to be understood in the following sense. Suppose a portion of air having the volume V to be slightly compressed so that its volume is reduced to $V-v$, where $\frac{v}{V}$ is a small fraction. Let P denote the pressure per unit of area exerted by the air before and $P+p$ after compression; then the quotient of p by $\frac{v}{V}$ is called the coefficient of elasticity.

99. The compression of air raises its temperature; and hence, if air is suddenly compressed, and then allowed to regain its original temperature, without further change of volume, its pressure immediately after compression is greater than that which it finally attains. But this final pressure is to its pressure before compression in the inverse ratio of the volumes, so that if this final pressure be $P + p$, we have

$$\frac{P+p}{P} = \frac{V}{V-v},$$

or

$$1 + \frac{p}{P} = \frac{1}{1-\frac{v}{V}} = 1 + \frac{v}{V} \text{ nearly},$$

that is, we have

$$\frac{p}{P} = \frac{v}{V},$$

The quotient of p by $\frac{v}{V}$ is therefore the same as the

quotient of p by $\frac{p}{\text{P}}$, that is to say, it is P. The coefficient of elasticity at constant temperature is therefore equal to the pressure.

In the compressions which accompany the propagation of sound through the air, the heat of compression has not time to escape, and hence the coefficient of elasticity, on which the velocity of sound depends, is not the coefficient for constant temperature, but is greater. Instead of being equal to P, it is about 1.41 P.

If the medium be any gas at pressure P, the coefficient of elasticity will be $(1 + \beta)$ P, where $1 + \beta$ is not very different from 1.41.

100. If we are careful to employ the same units consistently in our specification of all the quantities involved, the formula for the velocity of propagation of sonorous waves in air, in other words the velocity of sound, will be

$$v = \sqrt{\frac{1.41 \text{ P}}{\text{D}}},$$

v denoting this velocity (not *volume* as in the preceding section), and D denoting the density of the air.

101. Changes of barometric pressure do not affect the velocity of sound, for when the barometer rises P and D increase in the same ratio. But temperature does affect the velocity, for rise of temperature increases P if D is constant, or diminishes D if P is constant. Hence sound travels fastest when the air is warmest. Its velocity at

the mean temperature of this country is about 1,100 feet per second

102 When sound is propagated in open spaces, the sound-waves are of spherical form and become larger as they advance further from the source. Hence, as the amount of energy contained in them cannot increase, the same amount of energy is spread over a continually larger volume, and the intensity of the sound diminishes rapidly as the distance increases

When it is propagated through a speaking-tube, or other tube of uniform bore, there is a little communication of sonorous energy to the sides, but unless the tube is very long this portion is small, and the greater part of the energy is transmitted through the enclosed column of air. As the waves in this case do not increase in area, there is but little diminution of intensity. The velocity is the same through tubes as in the open air.

103. Sound is propagated through liquids in the same manner as through gases, and in most cases with greater velocity, for instance, the velocity in water is more than four times as great as in air. In both liquids and gases the propagation depends only on compression and expansion of volume, and the vibrations are longitudinal, that is, parallel to the direction in which the sound is propagated. But solids oppose resistance to change of shape as well as to change of size, and can transmit other kinds of vibrations besides those of longitudinal compression and extension, each kind having in general a different

velocity. Moreover, solids of a fibrous, laminated, or crystalline structure, have usually different velocities of propagation in different directions.

104. Elastic strings and wires afford examples of the propagation of *transverse* vibrations. If a long india-rubber cord fastened at the ends receives a smart lateral blow close to one end, a pulse is seen to run along it. If the point where the blow is delivered is remote from the ends, two pulses will be seen to start from this point and run along the cord in opposite directions. The ordinary

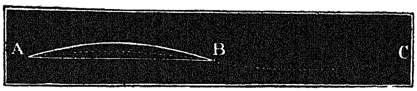

FIG 41

vibration of a musical string may be regarded as stationary undulation, and can be resolved into two sets of equal waves travelling in opposite directions. Thus, let the continuous curve in Fig. 41 represent a string at rest in a position of maximum displacement, in one plane, and let the dotted line be drawn midway between the string and the straight line A B, so that its ordinates are the halves of the corresponding ordinates of the string. Continue the dotted curve to C, making B C equal to A B, and making the continuation a reversed copy (reversed both up and down, and left and right) of the portion between A and B. Then the whole dotted line A B C represents one complete wave of either of the two component undulations which, by travelling with equal

velocities in opposite directions, would give, between the points A and B, the actual motion of the string. At these two points a positive ordinate of one component will always be compounded with an equal negative ordinate of the other, thus producing rest. The two component undulations travel with a velocity v, which, when consistent units are employed, is given by the formula

$$v = \sqrt{\frac{F}{m}},$$

where F denotes the force with which the string is stretched, and m the mass of unit length of the string. Since the wave-length A C is twice the length of the string, the period T is given by the equation

$$T = \frac{2l}{v}.$$

CHAPTER IX.

REFLECTION OF SONOROUS UNDULATIONS.

105. We have seen, in § 60, that two similar systems of simple waves travelling in opposite directions along the same column of air produce at certain sections of the column, called nodes, a mutual destruction of velocity combined with a double variation of density. A rigid diaphragm stretched across one of these sections would have no effect on the movements, for the particles of air at this section are permanently at rest whether the diaphragm be present or absent.

Conversely, a rigid diaphragm or stopper at one end of a pipe produces reflection of waves travelling along the pipe, and the reflected waves may be regarded as part of an imaginary system coming from the other side of the stopper, having such a relation to the incident system as jointly to produce a node at the surface of the stopper. The reflected waves must therefore be exact copies of the incident waves with velocities reversed.

106. Again, we have seen in § 61 that two such systems of opposite waves produce at certain other sections called antinodes a mutual destruction of effect as regards disturbance of density, combined with a double

velocity. The pressure at an antinode is the same as that of the undisturbed air; hence, if a hole be made in the side of the pipe at an antinode, there will be no tendency for air to pass through the hole either way, and the state of things within the pipe will remain unaffected. If the pipe be cut across at an antinode, and one of the two portions removed, the vibrations in the remaining portion will go on as before.

Conversely, when waves travelling along a pipe arrive at an open end, a state of things is produced in the pipe which is the same as if a system of waves were entering the pipe at this end, and producing jointly with the incident waves an antinode at the open end. The reflected waves must therefore in this case be copies of the incident waves with disturbance of density reversed. A pulse of compression will yield a reflected pulse of rarefaction, and a pulse of rarefaction will yield a reflected pulse of compression.

107. Echo is a familiar example of the reflection of sonorous undulations.

We may mention, as illustrating both kinds of reflection, that there is a well at Kentish Town, belonging to the New River Company, where an eight-inch iron pipe descends from a little above the ground to some hundreds of feet, and the water stands in it at a depth of rather more than 200 feet from the top. Words spoken into the mouth of this tube are very distinctly echoed from the surface of the water, and if spoken loudly they are

echoed more than once. A word loudly shouted is repeated about seven times, becoming gradually feebler with each repetition. The explanation is that the sonorous waves are reflected backwards and forwards, between the surface of the water below and the open end of the tube above. To produce one echo, they must travel once down the tube and up again ; to produce two echoes they must travel over twice this distance, and so on.

When the reflected waves reach the open end of the tube they are reflected down, and then again reflected up from the surface of the water.

108. Resonance is another example of the reflection of sonorous undulations.

When a vibrating tuning-fork is held at the mouth of a tube of proper length, the sound is greatly intensified by the resonance of the tube. If the tube is open at the far end, this effect will be obtained when its length is about half the wave-length of the note of the fork; for every pulse originated by the fork is reflected from the far end with reversal of condensations and extensions (which we shall call, for shortness, reversal of form), and after travelling back to the near end is again reflected with a second reversal, which restores it to its original form. If the time occupied in this process (that is, the time of travelling over twice the length of the tube) is equal to the period of vibration of the fork, the next pulse from the fork will exactly concur with the reflected pulse, and their amplitudes will be added. As each

original pulse gives rise to a long series of reflections, a great number of amplitudes will be added together, if the length of the tube is such as to make the coincidence of period exact.

If the tube is closed at the far end, the pulses will have to travel four times over its length in order to be restored by two reversals to their original form. The tube will therefore respond if its length is one-fourth of the wave length of the note emitted by the fork.

These are the shortest lengths that will suffice in the two cases. Resonance will also be obtained when the open tube is any multiple and the stopped tube any *odd* multiple of the shortest length, as will appear on tracing the successive reflections in each case.

109. Reflection such as we have here described takes place in organ pipes and wind instruments generally. From each end a reflected undulation is continually flowing through the pipe, and the combination of these two undulations travelling in opposite directions produces a stationary undulation, according to the principles of Chapter V. If the pipe is 'stopped,' there is a node at the stopped end; if it is open, there is an antinode at the open end; and in both cases there is an antinode at the end where the wind enters, which is always to a certain extent open.

The notes to which a pipe can respond are the same as those which it is fitted to yield. The lowest of these (which is the note that it is always made to yield in the

organ) is called its *first* or *fundamental tone.* The others are called its *overtones.* Their respective wave-lengths are most easily deduced from the following considerations.

1. At an open end there must always be an antinode, and at a stopped end a node.

2. The distance between a node and the nearest antinode is a quarter of a wave-length, and the distance between two consecutive nodes or two consecutive antinodes is therefore half a wave-length.

110. From these principles it follows that an open pipe must contain an even number, and a stopped pipe an odd number of quarter waves, so that if l denote the length of a pipe, and λ the wave-length of one of its tones, we have, for an open pipe,

$$l = \frac{2\lambda}{4}, \text{ or } \frac{4\lambda}{4}, \text{ or } \frac{6\lambda}{4}, \text{ \&c.,}$$

and for a stopped pipe

$$l = \frac{\lambda}{4}, \text{ or } \frac{3\lambda}{4}, \text{ or } \frac{5\lambda}{4}, \text{ \&c.}$$

From these values it is easy to show that the values of λ are proportional to 1, $\frac{1}{2}$, $\frac{1}{3}$, &c. for an open pipe, and to 1, $\frac{1}{3}$, $\frac{1}{5}$, &c. for a stopped pipe; whence it follows that the number of vibrations per second is proportional to 1, 2, 3, &c. for an open pipe, and to 1, 3, 5, &c. for a stopped pipe.

111. These statements would be exact if the air in the pipe vibrated in parallel plane layers, so that the motion

of all particles in the same cross section was the same, and was parallel to the length of the pipe. The actual wave-lengths are rather greater, and the actual numbers of vibrations consequently rather less than the above calculations would make them. The pitch of the overtones is more affected by this correction than the pitch of the fundamental; so that, for example, the second tone of an open organ-pipe (especially if the pipe is wide in proportion to its length) has not quite double the number of vibrations of the first.

112. The overtones of a musical string follow the same laws as those of an open organ-pipe.

When a pulse, consisting of a protuberance on one side of a string, runs along it, the particles of the string are drawn to this side as the protuberance reaches them, and return to their original position as it leaves them and passes on. On its arrival at one of the fixed ends of the string, it is unable to draw the fixed support to one side, and the additional resistance produces a rebound, throwing the protuberance over to the other side, and starting a reversed pulse, which travels along the string from this end to the other, where it is again reflected and reversed. The two portions of Fig. 42 will explain what is here meant. One of them (it is immaterial which) shows the original, and the other the reflected pulse. Wherever we suppose the pulse to be at a given moment, it will have travelled over twice the length of the string before it comes back to its original position and circumstances.

If we call the ends A and B, the pulse first travels from its

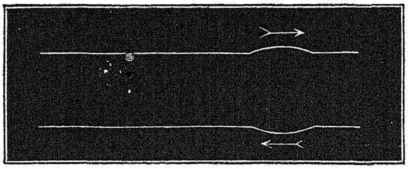

FIG. 42.

original position to B, thence in reversed form to A, and thence in its original form to its original position.

The stationary vibration of a string may be regarded (see § 104) as the resultant of two equal undulations travelling in opposite directions, their common wave-length being such as to give a node at each end of the string. The length of the string is therefore half a wave-length, or a multiple of half a wave-length. If l denote the length of the string (between the fixed supports), and λ the wave-length of the undulations which are propagated along it, we have

$$l = \frac{m\lambda}{2},$$

m being any integer.

If v denote the velocity of propagation along the string, the number of vibrations per second will be

$$\frac{v}{\lambda}, \text{ or } \frac{vm}{2l},$$

the successive values of which are

$$\frac{v}{2l}, \quad \frac{2v}{2l}, \quad \frac{3v}{2l}, \text{ \&c.}$$

The undulations produced in the air by the action of

the string will have the same periodic time (and therefore the same number of vibrations per second) as the string itself, but will have a different wave-length unless the velocity of propagation v along the string happens to be equal to the velocity of sound in air.

113. In giving its first or fundamental tone, the string passes backwards and forwards between the two positions shown by the continuous and dotted lines in Fig. 43.

In giving its second tone, its two extreme positions are those indicated by the continuous and the dotted line

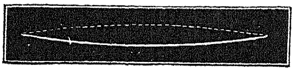

FIG. 43

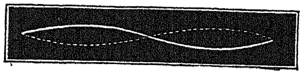

FIG 44

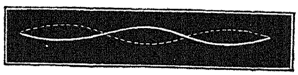

FIG 45

in Fig. 44. There is here a node in the centre besides the two fixed points at the ends.

In like manner, for its third tone, it divides into three equal parts separated by two nodes as in Fig. 45; and higher tones are in like manner obtained by carrying the division further.

If the string is forcibly started in any one of these modes of vibration and then left to itself, it will continue to vibrate in the same manner, the nodes remaining at the same places, but the amplitudes gradually becoming smaller. Hence these are called modes of free vibration of a string. A *mode of free vibration* for any body is a

mode of vibration which the body can maintain of itself when once started.

Modes of vibration resembling those here described can be started in a string by lightly touching it at one of the points where a node is required, while a fiddle-bow is drawn across it at a place where a node is not required. As many as eight or twelve successive tones can thus be elicited from an ordinary fiddle-string or piece of pianoforte wire of suitable length. A piece of wire stretched upon a sounding-box is sold, under the name of a *sonometer* or *monochord*, by makers of acoustic apparatus.

114. All stringed instruments have a sounding-box or board, for the purpose of communicating the vibrations of the strings to the surrounding air. A string stretched between two massive and firmly fixed blocks would give but a very faint sound; for the very small surface of the string itself is too small to enable it to produce powerful undulations in the air. In the violin and piano the string or wire is stretched over a bridge supported by a board (called in the piano the *sounding-board*); and it is the vibration of this board, with its large surface, that has the principal share in communicating disturbance to the air. In the violin the *belly*, on which the bridge rests, transmits its vibrations to the *back* with the help of the *sound-post*, and thus both the belly and the back act as sounding-boards. The agitation of the strings by the bow rocks the bridge from side to side, throwing pressure on its two feet alternately, and causing the two sounding-boards to vibrate normally.

CHAPTER X.

DYNAMICAL INVESTIGATION.

115. We shall now show that the elastic force of a stretched string is competent to produce such motion as we have been describing.

This motion is specified by the equation

$$y = \text{A} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi t}{\text{T}}, \qquad (18)$$

which is obviously the same as equation (15) of § 62, the amplitude of the stationary vibration at the points of maximum amplitude being now denoted by A instead of by $2a$.

The velocity $\frac{dy}{dt}$ and the acceleration $\frac{d^2y}{dt^2}$ of the particle x are,

$$\frac{dy}{dt} = -\frac{2\pi}{\text{T}} \text{A} \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{\text{T}},$$

$$\frac{d^2y}{dt^2} = -\left(\frac{2\pi}{\text{T}}\right)^2 \text{A} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi t}{\text{T}}$$

$$= -\left(\frac{2\pi}{\text{T}}\right)^2 y;$$

and our present business is to show that the forces of elasticity will produce this acceleration, if the tension F of

the string, and its mass per unit length m, satisfy the condition

$$\sqrt{\frac{F}{m}} = \frac{\lambda}{T}, \qquad (19)$$

the inclination θ of the string to the axis of x being supposed to be everywhere so small that its square is negligible.

Since $\cos\theta$ is $1-\frac{1}{2}\theta^2+$ higher terms, and $\sin\theta$ is $\theta-\frac{1}{6}\theta^3$ + higher terms, we may write $\cos\theta=1$, $\sin\theta=\theta$, and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \theta$.

The component tension parallel to the axis of x at any point is $F\cos\theta=F$, and the component tension parallel to the axis of y is $F\sin\theta = F\tan\theta = F\dfrac{dy}{dx}$; since in any plane curve $\dfrac{dy}{dx}$ expresses the tangent of the inclination of the curve to the axis of x.

The tension F being supposed the same at all points, the component parallel to the axis of x will therefore be the same, and in computing the resultant force acting on an element dx of the string, we may leave the components in this direction out of account. The component normal to the axis of x at the point x of the string is

$$F\frac{dy}{dx} = -F\frac{2\pi}{\lambda}A\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{T},$$

and the corresponding component at the point $x+dx$ is greater than this by the amount

$$-\mathrm{F}\frac{2\pi}{\lambda}\mathrm{A}\frac{2\pi}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{\mathrm{T}}dx$$

$$=-\mathrm{F}\left(\frac{2\pi}{\lambda}\right)^2\mathrm{A}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{\mathrm{T}}dx$$

$$=-\mathrm{F}\left(\frac{2\pi}{\lambda}\right)^2 y\,dx.$$

This is the resultant force upon the element dx, and the acceleration will be found by dividing by the mass $m\,dx$. The acceleration will therefore be

$$-\frac{\mathrm{F}}{m}\left(\frac{2\pi}{\lambda}\right)^2 y.$$

Substituting for $\frac{\mathrm{F}}{m}$ its value $\left(\frac{\lambda}{\mathrm{T}}\right)^2$, this expression becomes $-\left(\frac{2\pi}{\mathrm{T}}\right)^2 y$, which was to be proved.

116. Very similar reasoning applies to the stationary undulation of a cylindrical column of air. Let the motion of the particles of air be specified by the equation

$$y=\mathrm{A}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{\mathrm{T}}, \qquad (18)$$

y denoting the longitudinal displacement of the particle whose undisturbed position was x, so that $x+y$ is its actual position at time t. Then $\frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$ will still denote respectively the velocity and acceleration of the particle, and we have to show that the value of $\frac{d^2y}{dt^2}$, as deduced from equation (18), namely

$$\frac{d^2y}{dt^2}=-\left(\frac{2\pi}{\mathrm{T}}\right)^2 y,$$

is precisely the acceleration due to the elastic force of the air, if the coefficient of elasticity E (see § 98) and the density D fulfil the relation

$$\sqrt{\frac{\mathrm{E}}{\mathrm{D}}} = \frac{\lambda}{\mathrm{T}},$$

the compressions and extensions of the air, as measured by the ratio $\frac{v}{\mathrm{V}}$ of § 98, or by $\frac{dy}{dx}$ in our present notation (see § 51), being everywhere so small that their squares are negligible.

Let P be the undisturbed pressure. Then the actual pressure at time t, at the particle whose undisturbed co-ordinate was x, is

$$\mathrm{P} - \mathrm{E}\frac{dy}{dx},$$

that is

$$\mathrm{P} + \mathrm{E}\frac{2\pi}{\lambda}\mathrm{A}\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{\mathrm{T}}.$$

At the particle whose undisturbed co-ordinate was $x + dx$, the pressure is given by this expression, together with the additional term

$$\mathrm{E}\left(\frac{2\pi}{\lambda}\right)^2 \mathrm{A}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi t}{\mathrm{T}}\,dx,$$

or

$$\mathrm{E}\left(\frac{2\pi}{\lambda}\right)^2 y\,dx.$$

A layer of air of original thickness dx, of unit area and of original density D, is therefore subjected on its two faces to two opposite forces whose resultant is a backward force

$$\mathrm{E}\left(\frac{2\pi}{\lambda}\right)^2 y\,dx,$$

or a forward force

$$-\mathrm{E}\left(\frac{2\pi}{\lambda}\right)^2 y\,dx,$$

and dividing by the mass of the layer, which is $\mathrm{D}\,dx$, we find for the acceleration the value

$$-\frac{\mathrm{E}}{\mathrm{D}}\left(\frac{2\pi}{\lambda}\right)^2 y,$$

which, when we replace $\frac{\mathrm{E}}{\mathrm{D}}$ by its value $\left(\frac{\lambda}{\mathrm{T}}\right)^2$, becomes

$$-\left(\frac{2\pi}{\mathrm{T}}\right)^2 y,$$

as was to be proved.

117. In general, for the propagation of any disturbance along a cylindrical column of air, we have

$$\mathrm{P}-\mathrm{E}\,\frac{dy}{dx}$$

as the expression for the pressure of the particle of air whose undisturbed ordinate was x The expression for the pressure at the particle whose undisturbed ordinate was $x+dx$ is

$$\mathrm{P}-\mathrm{E}\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\,dx\right).$$

Hence the pressure in front of a layer of original thickness dx exceeds the pressure behind it by

$$-\mathrm{E}\,\frac{d^2y}{dx^2}\,dx,$$

or the pressure behind exceeds the pressure in front by

$$\mathrm{E}\frac{d^2y}{dx^2}\,dx.$$

As the mass of the layer per unit area is D dx, the acceleration is

$$\frac{\mathrm{E}}{\mathrm{D}}\frac{d^2y}{dx^2}.$$

But the acceleration is denoted by $\frac{d^2y}{dt^2}$. Hence we must have

$$\frac{d^2y}{dt^2}=\frac{\mathrm{E}}{\mathrm{D}}\frac{d^2y}{dx^2}. \tag{20}$$

118. This condition is satisfied by the equation

$$y=a\cos\frac{2\pi}{\lambda}(x-vt),$$

and also by the equation

$$y=a\cos\frac{2\pi}{\lambda}(x+vt),$$

provided that in both cases

$$v=\sqrt{\frac{\mathrm{E}}{\mathrm{D}}};$$

for either of these equations gives

$$\frac{d^2y}{dt^2}=-\left(\frac{2\pi v}{\lambda}\right)^2y,$$

$$\frac{d^2y}{dx^2}=-\left(\frac{2\pi}{\lambda}\right)^2y,$$

whence

$$\frac{d^2y}{dt^2}=v^2\frac{d^2y}{dx^2},$$

Similar reasoning will apply to the propagation of transverse waves along a string.

It thus appears that the undulations which we have been discussing in the previous chapters from a merely kinematical point of view, are dynamically possible as consequences of the laws of elasticity.

CHAPTER XI.

ENERGY OF VIBRATIONS.

119. An ordinary pendulum affords a very good example of the transformation of energy. When we draw it aside from its lowest position we do work against gravity, the amount of this work being equal to the weight of the pendulum multiplied by the height of the new position of its centre of gravity above its lowest position. If we now release the pendulum it falls back, and in the fall gravity does as much work upon it as was previously done against gravity. This amount of work may be called the energy of vibration of the pendulum, and it is continually undergoing transformation. In the two extreme positions it is all in the shape of 'potential energy,' or as it may be better called 'statical energy,' because its amount is computed without any reference to the laws of motion. In the lowest position it is all in the form of 'kinetic energy' or energy of motion, the amount of which is measured by multiplying each element of the mass by half the square of its velocity and adding these products for the whole pendulum. In intermediate positions the energy is partly in the one form and partly in

the other. The statical energy in any position is computed by multiplying the weight of the pendulum by the height of its centre of gravity above the lowest position of the centre of gravity, and is continually diminishing as the pendulum descends, while the kinetic energy undergoes a corresponding increase. In the ascent a converse change takes place, and the total amount (computed by taking the sum of the two energies) is the same in all positions.

120. Let the pendulum be a 'simple pendulum,' consisting of a mass m suspended by a string of length l without weight or mass, and let α be the angle which the string in the extreme positions makes with the vertical. Then the statical energy in the extreme positions is

$$mg\ l \text{ vers } \alpha.$$

The velocity which the mass m will acquire in descending through the vertical distance l *vers* α to its lowest position is given by the formula

$$v^2 = 2g\ l \text{ vers } \alpha.$$

Hence the value of $\frac{1}{2}\ m\ v^2$, or the energy of motion, in the lowest position, is

$$\tfrac{1}{2}\ mv^2 = m\ gl \text{ vers } \alpha,$$

which is the same as the statical energy in the extreme positions.

In any intermediate position, let θ be the inclination of the string to the vertical. Then the statical energy is

$$m\ gl \text{ vers } \theta,$$

and the kinetic energy is $\frac{1}{2}mv^2$, where

$$v^2 = 2gl\,(vers\ \alpha - vers\ \theta),$$

giving $\frac{1}{2}mv^2 = mgl\,(vers\ \alpha - vers\ \theta)$,
which, added to the statical energy, gives for the total energy $mgl\ vers\ \alpha$, as before.

121. When the angle α is small, the total energy is

$$mgl\ vers\ \alpha = 2mgl\sin^2\frac{\alpha}{2} = 2mgl\left(\frac{\alpha}{2}\right)^2 = \frac{1}{2}mgl\,\alpha^2,$$

which is proportional to the square of the amplitude, $l\alpha$ being the amplitude. Also, the effective force on the mass m in any position—that is, the component force along the tangent—is $mg\sin\theta$, which increases from zero in the lowest position to $mg\sin\alpha$ in the extreme positions. The distance through which this effective force works during the movement from an extreme position to the lowest is $l\alpha$, and if we multiply this by half the maximum force, that is, by $\frac{1}{2}mg\sin\alpha$, which is the same as $\frac{1}{2}mg\,\alpha$, we obtain $\frac{1}{2}m\,gl\,\alpha^2$, which is identical with the expression above obtained for the whole work done or whole energy. This rule always holds for vibrations in which the force called out follows Hooke's law, that is to say, the mean working force (which if multiplied by the whole displacement gives the work done) is just half the maximum force. This can be proved as follows.

Let O (Fig. 46) be the position of equilibrium, A the extreme position, B and C two points equally distant from O and A respectively. Then if μ. OA be the expression for the force at A, the forces at B and C will be μ. OB

and at the moment under consideration $\sin \frac{2\pi t}{T}$ has its maximum value, namely unity.

$$\text{Hence, } v^2 = \left(\frac{4\pi a}{T} \cos \frac{2\pi x}{\lambda}\right)^2,$$

and we want to find the mean value of this expression as x increases by equal steps from 0 to $\frac{\lambda}{4}$, since the origin is at an antinode, and $\frac{\lambda}{4}$ is the distance from an antinode to a node. As x increases by equal steps from 0 to $\frac{\lambda}{4}$, the angle $\frac{2\pi x}{\lambda}$ increases by equal steps from 0 to $\frac{\pi}{2}$, and we want the mean value of the square of its cosine, which is evidently the same as the mean value of the square of its sine, since the successive values of the cosine in the first quadrant are the same as those of the sine taken in backward order. We may therefore write

$$\begin{aligned} \text{Mean of } (\text{cosine})^2 &= \text{mean of } (\text{sine})^2 \\ &= \tfrac{1}{2} \text{ mean of } (\cos^2 + \sin^2) = \tfrac{1}{2}, \end{aligned}$$

since $\cos^2 + \sin^2 = 1$.

The mean value of v^2 is therefore $\frac{1}{2}\left(\frac{4\pi a}{T}\right)^2$, and the mean value of $\frac{1}{2} v^2$ is $\left(\frac{2\pi a}{T}\right)^2$. The energy of vibration of the string is therefore equal to the mass of the string

multiplied by $\left(\frac{2\pi a}{T}\right)^2$, where it is to be observed that $2a$ denotes the amplitude of vibration at an antinode.

125. The foregoing reasoning applies equally to longitudinal vibration, and shows that the energy of the sonorous vibration of the air within a cylindrical pipe, when executing stationary vibration parallel to the length of the pipe, is equal to the mass of this air multiplied by $\left(\frac{2\pi a}{T}\right)^2$, where $2a$ still denotes the amplitude of vibration at an antinode.

126. We shall now investigate the energy of each of the two equal travelling undulations into which stationary undulation can be resolved.

If we employ the expression for one of these undulations (equation (10), § 49)

$$y = a \cos \frac{2\pi (vt - x)}{\lambda},$$

we have, for the velocity,

$$\frac{dy}{dt} = -\frac{2\pi v a}{\lambda} \sin \frac{2\pi (vt - x)}{\lambda}.$$

The mean value of the square of this velocity is $\left(\frac{2\pi v a}{\lambda}\right)^2$ multiplied by the mean value of $\sin^2 \theta$, where θ denotes $\frac{2\pi (vt - x)}{\lambda}$, and we may either make t constant, and thus take the mean at a given moment along a wave-length, or make x constant, and take the mean at a given point for

the period of one vibration. In either case we shall have to take the mean value of $\sin^2\theta$ for an entire circle, which is the same as its mean value for the first quadrant, and is $\frac{1}{2}$, as shown in § 124. The mean value of (velocity)^2 is therefore

$$\frac{1}{2}\left(\frac{2\pi\,v\,a}{\lambda}\right)^2,\ \text{or, since } \frac{v}{\lambda}=\frac{1}{T}$$

$$\frac{1}{2}\left(\frac{2\pi\,a}{T}\right)^2,$$

and the kinetic energy is the mass multiplied by the half of this, that is

$$\text{mass} \times \left(\frac{\pi\,a}{T}\right)^2.$$

The amount of the statical energy can be deduced from the fact that the wave now under consideration has half the amplitude of the stationary wave. The energy of the stationary wave of amplitude $2a$ is entirely statical at the moment of extreme displacement, and is, as we have seen,

$$\text{mass} \times \left(\frac{2\pi\,a}{T}\right)^2.$$

Hence the statical energy of the travelling wave of amplitude a is

$$\text{mass} \times \left(\frac{\pi\,a}{T}\right)^2,$$

and is equal to its kinetic energy. The total energy of the travelling wave is

$$\text{mass} \times 2\left(\frac{\pi\,a}{T}\right)^2,$$

and is half the energy of the stationary wave, as we

should have expected from the circumstance that the stationary wave is resolvable into two travelling waves.

127. The travelling waves which combine to form stationary waves are in opposite directions. When two systems of waves travelling in the same direction, of approximately the same wave length (both longitudinal, or both in the same plane if transverse), are compounded, we have, by § 33, at any instant,

$$c^2 = a^2 + b^2 + 2ab \cos \theta,$$

c denoting the resultant amplitude of any particle, a and b the amplitudes of the two components, and θ their difference of phase at the instant considered. Hence, as shown in § 35, the mean value of c^2 is $a^2 + b^2$, whence it can be shown that the energy of the resultant system is the sum of the energies of the two components. If a and b are equal, the values of c^2 will range between zero and $4a^2$, so that the smallest resultant waves will have no energy (being in fact evanescent), and the largest will have four times the energy of a wave of either component.

128. When waves of sound spread out uniformly in all directions from a centre, they form spheres which are continually enlarging. As each wave carries with it its original amount of energy unchanged, the same amount of vibratory energy is propagated across all the imaginary spherical boundaries which can be described about the centre. But the surfaces of these spheres are propor-

tional to the squares of their radii, and hence the amounts of energy propagated across *equal areas* of two of the spheres are inversely as the squares of their radii. The intensity of sonorous vibration at a point, being measured by the quantity of energy which crosses unit area around the point in unit time, is therefore inversely proportional to the square of the distance of the point from the source, the source being supposed small in comparison with the distance. The amplitude of the sonorous vibrations will be inversely as the distance, since the energy of simple harmonic vibration is as the square of the amplitude.

129. When two sounds are alike, both in pitch and quality, their loudness is naturally measured by the energy of the sonorous vibrations which they excite. If they differ in pitch but agree in quality, and both lie within the usual compass of music, the sensation excited by the sound of higher pitch will in general be the more intense, if the energies are equal. This is proved by the fact that the bass pipes of an organ require a great deal more wind, and therefore more work in blowing, than the treble pipes.

Again, sounds of piercing quality strike the ear more powerfully than sounds of smooth quality containing the same amount of energy. A piercing quality is usually due to the presence of high harmonics.

CHAPTER XII.

SIMPLE AND COMPOUND TONES.

130. It was discovered by Ohm that a simple harmonic vibratory motion produces the sensation of a simple tone, and that when several simple tones are heard together each of them is due to its own simple harmonic component, which is present in the resultant sonorous vibration. Those musical tones which we call *rich* are not simple. The sound produced by striking one of the keys of a pianoforte is usually composed of some four or five simple tones, due to the co-existence in the wire of so many different modes of simple harmonic vibration. The tones of a violin are still more highly compound. We have pointed out in § 112 that the periods of the several modes of simple harmonic vibration of a string are proportional to 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., or, what amounts to the same thing, that the numbers of vibrations made in a given time are proportional to the series of natural numbers 1, 2, 3, 4, &c. When the partial tones which together compose a compound tone have this relation to one another, the deepest of them—that which corresponds to the number 1—is called the *fundamental*,

and the others are called *harmonics* of the fundamental. These terms are applied both to the partial tones themselves and to the simple harmonic component vibrations to which they are due. It was proved mathematically by Fourier that every periodic vibration—that is, every vibratory motion which exactly repeats itself in a definite period—can be resolved into a fundamental simple harmonic vibration and its harmonics. Over against this mathematical fact we may set the acoustic fact that every musical tone of definite pitch is composed of a simple tone of this pitch and its harmonics. The connection between these two facts is explained by Ohm's principle above mentioned.

131. The human ear is not by any means the only instrument that picks out the simple harmonic constituents from a compound vibratory motion. Strings mounted on a sounding-board, as in the pianoforte, will do the same thing. When the pedal is depressed, so as to remove the dampers from the wires, if a compound tone is sung or otherwise sounded in its neighbourhood, those strings which correspond to the partial tones present in the sound will be thrown into vibration, and thus the piano will echo back a compound sound very similar in its constitution to the original. To ascertain whether a given tone is present in the original sound, the pedal should not be depressed, but the key corresponding to this particular tone should be held down. In general, any body which can vibrate freely in one definite

period will be set in vibration if acted on by a periodic movement which, when analysed into simple harmonic constituents, contains one whose period agrees with that of the body in question.

132. Helmholtz's resonators, one of which is represented in Fig. 48, are intended to assist the ear in detecting the presence or absence of a given elementary tone in a compound sound. They are hollow globes of brass, with two openings; the smaller one is to be applied to the observer's ear, while the larger one is directed towards the source of sound. Each resonator corresponds to one definite simple tone, whose period is the same as the natural period of vibration of the body of air enclosed within the globe; and when this tone is present the resonance of the enclosed air produces a great increase in its intensity, so that the observer hears the resonator speak into his ear. These resonators are usually supplied in a series of ten, corresponding to the bass C of a man's voice and its first nine harmonics. When the bass C is sung, all or nearly all the resonators are observed to respond, thus proving the composite character of the tones of the human voice.

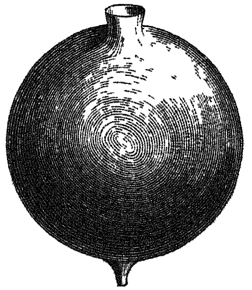

FIG. 48.

133. The same principles are illustrated, synthetically instead of analytically, in the 'mixture' stop of organs. When this stop is employed in conjunction with those

called 'principal' and 'diapason,' the pipes which are brought into use are so combined that on putting down any key the corresponding note and a series of its harmonics are all sounded at once, each by a separate pipe. The effect, as judged by an ordinary ear, is that of a single rich note of the pitch of the fundamental.

CHAPTER XIII.

MUSICAL INTERVALS.

134. WHAT is called in music the *pitch* of a note depends upon the period of vibration, or upon the number of vibrations in a given time. A note of high pitch is a note of short period, or of a large number of vibrations per second.

In comparing two notes, the *interval* between them depends only on the ratio of the two periods, or (the reciprocal of this) the ratio of the numbers of vibrations made in a given time. When the ratio is that of 1 : 2 the interval is called an *octave*; when it is 2 : 3 the interval is called a *fifth*; when it is 3 : 4 it is called a *fourth*; when it is 4 : 5 it is called a *third*, and so on. The origin of these names is due to the fact that, in the notes of the ordinary major scale, the numbers of vibrations are proportional to the following numbers:

Do	Re	Mi	Fa	Sol	La	Si	Do
24	27	30	32	36	40	45	48,

or, dividing by 24, to

$$1 \quad \tfrac{9}{8} \quad \tfrac{5}{4} \quad \tfrac{4}{3} \quad \tfrac{3}{2} \quad \tfrac{5}{3} \quad \tfrac{15}{8} \quad 2.$$

The interval $\frac{9}{8}$ is called a *second*, $\frac{5}{4}$ a *third*, $\frac{4}{3}$ a *fourth*, and so on.

135. If we examine the intervals between the successive notes of the above scale, we find them to be as follows:—

$$\frac{9}{8} \quad \frac{10}{9} \quad \frac{16}{15} \quad \frac{9}{8} \quad \frac{10}{9} \quad \frac{9}{8} \quad \frac{16}{15}.$$

The pitch of the note Do, which is called the *key-note*, may be anything we please, and in vocal music it is possible to conform exactly to the above scale, whatever the pitch may be. But the majority of musical instruments are limited to a definite number of notes (12 for each octave, or 13 if we include both the initial and terminal note), and to accommodate music to their requirements a compromise is made, which is called *temperament*. There are various systems of temperament; but the one which is now most generally approved is that which is called the system of *equal temperament*, because it is equally inexact for all key-notes. In this system the intervals represented by $\frac{9}{8}$ and $\frac{10}{9}$ in the above scale are replaced by the sixth root of 2, and the intervals $\frac{16}{15}$ bythe twelfth root of 2, so that twelve of these last intervals or six of the former would make an octave.

If we employ the logarithms of the numbers of vibrations instead of the numbers themselves, we shall be able to get a better idea of the relative magnitudes of the intervals, for equal differences between the logarithms will

correspond to equal ratios between the numbers, and therefore to equal intervals between the notes.

The logarithm of $\frac{9}{8}$ is ·051 nearly.

,,	,,	$\frac{10}{9}$,,	·046	,,
,,	,,	$\frac{16}{15}$,,	·028	,,
,,	,,	$\sqrt[6]{2}$,,	·050	,,
,,	,,	$\sqrt[12]{2}$,,	·025	,,

136. We are now in a position to give the musical values of the harmonics of a note. Let the fundamental be called Do_1, its octave Do_2, its double octave Do_3, and so on, then the following table exhibits the values of the successive harmonics up to that which has ten times the number of vibrations of the fundamental.

Number of vibrations.	Musical values.
1	Do_1
2	Do_2
3	Sol_2
4	Do_3
5	Mi_3
6	Sol_3
7	not exact.
8	Do_4
9	Re_4
10	Mi_4

137. As regards absolute pitch, the middle C of a pianoforte, which is at the top of a bass voice, and at the bottom of a treble voice, has 256 vibrations per second

according to what is called the theoretical pitch, which is the pitch adopted by the best makers of acoustic apparatus. The pitch usually adopted by musicians in this country is rather higher—from 264 to 270 vibrations per second for the middle C. In the 'theoretical pitch,' the number of vibrations for any octave of C is a power of 2; for example, 256 is the eighth power of 2.

138. Some combinations of sounds produce a pleasing sensation and are called concords; others produce an unpleasant sensation and are called discords; while others again are intermediate in quality.

If we take two organ pipes whose pitches we can adjust, we find that if we tune them first to unison, and then gradually increase the difference between them to a semitone, they begin to 'beat' as soon as they cease to be in unison, and the rapidity of the beats increases with the interval, until it produces a rattling or jarring sound which is very unpleasant. If one makes 256 and the other 240 vibrations in a second, the number of beats in a second will be 16, being the difference of these numbers.

If, instead of the pipes being initially in unison, one is the octave of the other, we shall still hear beats when we put them out of tune, though not so loud as in the former case. If one of them makes 250 and the other 512 vibrations per second, the number of beats per second will be 12, being the difference between 512 and the double of 250. If we listen attentively, aiding the

ear, if necessary, with two resonators, one responding to the lower and the other to the upper note, we shall find that the lower note keeps steady, and that it is the upper note only that beats. In fact, the note emitted by the lower pipe is not a simple tone, but contains harmonics, the loudest of which has 500 vibrations per second, and this harmonic beats with the fundamental tone of the upper pipe. All beats arise, as in these two instances, from two simple tones in approximate unison. The ear is believed to contain some thousands of vibrating fibres, each having its own natural period of vibration and being thrown into vibration by simple tones whose period agrees or nearly agrees with this. The greater the departure from exact agreement the feebler will be the effect. Thus every simple tone that reaches the ear excites a select group of fibres, the middle members of the group (as regards their natural period) being excited strongly, and the extreme members very feebly. When two simple tones nearly in unison reach the ear together, the two groups of fibres which are excited will overlap, that is to say, some fibres will be common to both, and the vibrations of these fibres, being the resultant of two simple harmonic motions of nearly equal period, will vary in amplitude from the sum to the difference of the amplitudes of the two components. These fibres therefore beat, and thus it is that we hear beats.

Discord, according to Helmholtz, whose theory is now generally accepted by writers on acoustics, is always

due to beats, and the harshness of a given combination depends partly upon the strength of the beats and partly upon their rapidity. Very slow beats are not unpleasant, and very rapid beats may be so rapid as to escape observation in virtue of the persistence of impressions. The rapidity which produces a maximum of unpleasantness depends partly on the pitch, being greater as the pitch is higher, and may be roughly stated as being from $\frac{1}{10}$th to $\frac{1}{20}$th of the rapidity of the vibrations themselves.